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**CHAPTER I**

**PREREQUISITES**

* 1. **Introduction.**

Topology is main branch of pure Mathematics the purpose of subject is to elucidate and inspect the concept of topological spaces, their continuity and continuos mapping, within the structure of Mathematics. The study of these and their general properties leads to a formation of general topology. The fundamental structure on a topological space is not a distance function, but a collection of open sets, thinking directly in terms of open sets often leads to greater clarity as well as greater generality. Here we present an detailed study of a new kind of generalized closed sets termed ws-closed sets and their respective continuous maps, closed maps, open maps, homeomorphisms, locally closed sets, locally continuous maps and bitopological spaces.

This thesis includes an overview on mathematicans and their work on toplogical spaces for the progress of topology. Second section discussion starts with stronger and weaker forms of open sets and closed sets. In third Section deals with stronger and weaker forms of continuous functions and irresolute maps. Section 4 deals with some closed maps, open maps, section 5 elaborates homeomorphisms. Section 6 explains the concepts of locally closed sets, furher LC-continuous maps. In section 7 bitopological spaces and in section 8 speration of axioms are explained.

Complete in thesis P, Q and R denote the topological spaces (P,τ), (Q,σ) and (R, η) respectively for which separation axioms are not assumed untill mentioned explicitly. For each subset D of a space (P, τ), the closure of D, interior of D, semi-interior of D, semi-closure of D, w-interior of D, ws-closure of D, and the complement of D

are denoted by cl(D) or τ-cl(D), int(D) or τ-int(D), sint(D), scl(D), ws-int(D), ws-cl(D), and DC or P−D respectively.

**1.2 Stronger and weaker forms of open sets and closed sets.**

Stone [88], Tong [91] and Velicko [94] have popularized and investigated stronger type open sets termed as regular, strong regular open sets respectively. Levine [50], Bhattachary and Lahiri [15], Mashhour [60], Biswas [16], Gnanambal [39], Veera Kumar [93], Palanippan and Rao [72], Sheik Jhon [84] and Nagaveni [65] have respectively determined generalized , semi-generalized, α, semi, g\*, regular generalized, w and weakly generalized closed sets. The complements of these different types of open (closed) sets are called the same type of closed (open) sets and so on.. The following definitions are the prerequisitesfor the present study.

**Definition1.2.1:** For subset D of (P, τ), if D ⊆ cl (int (D)) then it is semi open set and

If int (cl(D)) ⊆ D then it is semi closed set.

**Definition1.2.2**: For subset D of (P, τ), if D ⊆ int (cl (D)) then it is pre-open set (1982) and if cl (int (D)) ⊆ D then it is pre-closed set.

**Definition 1.2.3**: For subset D of (P, τ),if D ⊆ int (cl(int(D))) then it is α-open set (1965) and if cl(int(cl(D)))⊆ D then it is α -closed set.

**Definition 1.2.4:** For subset D of (P, τ), if D ⊆ cl (int (cl (D)))) then it is semi-pre-open set (1986) (β-open [1]) and if int (cl (int (D))) ⊆D then it is a semi-pre closed set (β-closed). **Definition 1.2.5:** For a subset D of (P, τ), if D = int (cl (D)) then it is regular open set (1937) and if D = cl (int (D)) then it is a regular closed set.

**Definition 1.2.6:** For subset D of (P, τ), if there is a regular open set P such that P ⊆ D ⊆ αcl(P) then it is regular α-openset (2009) (briefly, rα-open).

**Definition 1.2.7:** For subset D of (P, τ), if there is a regular open set P such that P ⊆ D ⊆ cl(P) then it is regular semi openset (1978).

**Definition 1.2.8:** For subset D of a topological space (P, )

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Sl.no** | **Name of the set** | **if** | **whenever** | **M is** |
| 1 | Generalized closed set (g-closed)[35] | cl(D) ⊆ M | D ⊆ M | open in (P, τ). |
| 2 | Generalized semi closed set (gs-closed)[7] | scl(D) ⊆ M | D ⊆ M | open in (P, τ) |
| 3 | Semi generalized closed set (sg-closed) [13] | scl(D) ⊆ M | D ⊆ M | semi open in (P, τ). |
| 4 | Generalized semi pre closed set (gsp-closed) [22] | spcl(D) ⊆ M | D ⊆ M | open in (P, τ). |
| 5 | α-generalized closed set ( αg- closed)[37] | αcl(D) ⊆ M | D ⊆ M | open in (P, τ). |
| 6 | Generalized α-closed set (gα closed)[38] | αcl(D) ⊆ M | D ⊆ M | α-open in (P, τ). |
| 7 | Regular generalized closed set (rg-closed) [46] | cl(D) ⊆ M | D ⊆ M | Regular-open in (P, τ). |
| 8 | Generalized pre closed set (gp-closed)[39] | pcl(D) ⊆ M | D ⊆ M | open in (P, τ). |
| 9 | Generalized pre regular closed set (gpr-closed)[24] | pcl(D) ⊆ M | D ⊆ M | regular open in (P, τ). |
| 11 | w-closed set[59] | cl(D) ⊆ M | D ⊆ M | semi -open in (P, τ). |
| 13 | swg-closed set[42] | cl(int(D)) ⊆ M | D ⊆ M | semi-open in (P, τ). |
| 14 | rwg-closed set[42] | cl(int(D)) ⊆ M | D ⊆ M | regular -open in (P, τ). |
| 15 | rw-closed set[10] | cl(D) ⊆ M | D ⊆ M | regular semi -open in (P, τ). |
| 16 | R\*-closed set[30] | rcl(D) ⊆ M | D ⊆ M | regular semi -open in (P, τ). |
| 17 | rgw-closed set[54] | cl(int(D))⊆ M | D ⊆ M | regular semi-open in (P, τ). |
| 18 | Wgrα–closed set[32] | cl(int(D)) ⊆ M | D ⊆ M | regular α-open in (P, τ) |
| 19 | pgpr-closed set[6] | pcl(D)) ⊆ M | D ⊆ M | rg-open in (P, τ) |
| 20 | rps-closed set[60] | spcl(D) ⊆ M | D ⊆ M | rg-open in (P, τ). |
| 21 | gprw-closed set[55] | pcl(D) ⊆ M | D ⊆ M | regular semi -open in (P, τ). |
| 22 | αrw-closed set[76] | αcl(D) ⊆ M | D ⊆ M | rw-open in (P, τ). |
| 23 | gα\*\*-closed set[] | cl(D) ⊆int(cl(M)) | D ⊆ M | α -open in (P, τ). |
| 24 | -closed set[67] | scl(D) ⊆ M | D ⊆ M | sg -open in (P, τ). |
| 25 | g# -closed set [66] | cl(D) ⊆ M | D ⊆ M | M is αg- open in (P, τ). |
| 26 | αgp-closed set[26] | cl(D) ⊆ M | D ⊆ M | pre-open in (P, τ). |
| 27 | pgrα-closed set[14] | cl(D) ⊆ M | D ⊆ M | regular α-open in (P, τ). |
| 28 | βwg\*-closed set[18] | gcl(D) ⊆ M | D ⊆ M | β-open in (P, τ). |
| 29 | –closed set[73] | if cl(D) ⊆ M | D ⊆ M | gα - open in (P, τ). |
| 30 | -closed) set[74] | cl(D) ⊆ M | D ⊆ M | \*gα - open in (P, τ). |
| 31 | gαb-closed set[75] | bcl(D) ⊆ M | D ⊆ M | α-open in (P, τ). |
| 32 | sgb-closed set[28] | bcl(D) ⊆ M | D ⊆ M | semi -open in (P, τ). |
| 33 | rgb-closed set[40] | bcl(D) ⊆ M | D ⊆ M | regular-open in (P, τ). |
| 34 | rg\*b-closed set[27] | bcl(D) ⊆ M | D ⊆ M | rg-open in (P, τ). |
| 35 | Pre-semi –closed set[68] | spcl(D) ⊆ M | D ⊆ M | spcl(D) ⊆ M |
| 36 | r^g-closed set[57] | gcl(D) ⊆ M | D ⊆ M | regular -open in (P, τ). |
| 37 | -closed set[69] | cl(D) ⊆ M | D ⊆ M | semi -open in (P, τ) |
| 38 | #gs-closed set[29] | scl(D) ⊆ M | D ⊆ M | \*g-open in (P, τ). |
| 39 | -closed set[29] | cl(D) ⊆ M | D ⊆ M | #gs -open in (P, τ). |
| 41 | g#α-closed set[45] | αcl(D) ⊆ M | D ⊆ M | g-open in (P, τ). |
| 42 | αgs-closed set[52] | αcl(D) ⊆ M | D ⊆ M | semi -open in (P, τ). |
| 43 | g#s-closed set[71] | scl(D) ⊆ M | D ⊆ M | M is αg -open in (P, τ). |
| 47 | gb-closed set[1] | bcl(D) ⊆ M | D ⊆ M | M is -open in (P, τ). |
| 48 | rb-closed set[43] | cl(D) ⊆ M | D ⊆ M | b-open in (P, τ). |
| 49 | swg\*-closed set[41] | gcl(D) ⊆ M | D ⊆ M | semi -open in (P, τ). |
| 50 | gr-closed set[58] | rcl(D) ⊆ M | D ⊆ M | open in (P, τ). |
| 51 | βwg\*\*-closed set[62] | βwg\* cl(D) ⊆ M | D ⊆ M | regular-open in (P, τ). |
| 52 | αs**-**closed set[] | scl(D) ⊆ M | D ⊆ M | αgs -open in (P, τ). |
| 53 | wα-closed set[] | cl(int(D)) ⊆ M | D ⊆ M | αgs-open in (P, τ). |
| 54 | R-closed set[] | αcl(D) ⊆int(M) | D ⊆ M | w-open in (P, τ). |
| 55 | gα\*-closed set[38] | αcl(D) ⊆int(M) | D ⊆ M | α-open in (P, τ). |
| 56 | -closed set[] | cl(D) ⊆ M | D ⊆ M | sg -open in (P, τ). |
| 57 | \*gα-closed set[73] | αcl(D) ⊆ M | D ⊆ M | gα-open in (P, τ). |
| 58 | #gα-closed set[17] | αcl(D) ⊆ M | D ⊆ M | g#α -open in (P, τ). |
| 59 | gξ\* -closed set[33] | αcl(D) ⊆ M | D ⊆ M | #gα- open in (P, τ). |
| 60 | g#-pre- closed set[51] | pcl(D) ⊆ M | D ⊆ M | g#-open in (P, τ). |
| 61 | gps-closed set[53] | pcl(D) ⊆ M | D ⊆ M | semi open in (P, τ). |
| 62 | gspr-closed set[25] | spcl(D) ⊆ M | D ⊆ M | regular -open in (P, τ). |
| 63 | wg-closed set[42] | cl(int(D)) ⊆ M | D ⊆ M | open in (P, τ). |
| 64 | rgα-closed set[63] | αcl(D) ⊆ M | D ⊆ M | regular α-open in(P, τ). |
| 66 | wα-closed set[11] | αcl(D) ⊆ M | D ⊆ M | w-open in (P, τ). |
| 67 | gwα-closed set[12] | αcl(D) ⊆ M | D ⊆ M | wα -open in (P, τ) |

The compliment of the above mentioned definition closed sets are their open sets.

1.3 ws- Continuous maps and irresolute maps:

In general topology, the concept of continuous functions plays a very important role. This section deals with continuous maps and irresolute maps contributed by few topologist namely Arya and Gupta [6], Dontchev and Maki [31], Levine [49], Balachandran et al [13], Mashhour [60], Sundaram [90], Devi [22], Gnanambal[41], Palaniappan [72], Sheik John[84], Benchalli[ ] and R.S. Walli [ ] has explored regular continuous and completely , semi , g, α, sg, gs, gpr, rg, w- continuous. Irresolute maps were innovated and explored by Crossely and Hildebrand [20].

**1.3.1 Definition:** In a map h: (P, τ) →(Q, σ)

1. Pretend h–1(D) is r-closed in P for all closed subset D of Q then it is termed as regular-continuous(r-continuous) [3].
2. Pretend h–1(D) is regular closed in P for all closed subset D of Q then it is termed as completely–continuous [3].
3. Pretend h–1(D) is clopen (both open and closed) in P for all subset D of Q then it is termed as strongly–continuous [26].
4. Pretend h–1(D) is α-closed in P for all closed subset D of Q then it is termed as α–continuous [14].
5. Pretend h–1(D) is α-closed in P for all semi-closed subset D of Q then it is termed as strongly α–continuous [32].
6. Pretend h–1(D) is αg–closed in P for all closed subset D ofQ then it is termed as αg–continuous [19].
7. Pretend h–1(D) is wg–closed in P for all closed subset D of Q then it is termed as wg–continuous [23].
8. Pretend h–1(D) is rwg–closed in D for all closed subset D of Q then it is termed as rwg–continuous [23].
9. Pretend h–1(D) is gs–closed in P for all closed subset D of Q then it is termed as gs–continuous [4].
10. Pretend h–1(D) is gp–closed in P for all closed subset D of Q then it is termed as gp–continuous [20].
11. Pretend h–1(D) is gpr–closed in P for all closed subset D of Q then it is termed as gpr–continuous [12].
12. Pretend h–1(D) is αgr–closed in P for all closed subset D of Q then it is termed as αgr–continuous [30].
13. Pretend h–1(D) is ωα–closed in P for all closed subset D of Q then it is termed as wα–continuous [7].
14. Pretend h–1(D) is gspr–closed in P for all closed subset D of Q then it is termed as gspr–continuous [24].
15. Pretend h–1(D) is g–closed in P for all closed subset D of Q then it is termed as g–continuous [7].
16. Pretend h–1(D) is w-closed in P for all closed subset D of Q then it is termed as ω–continuous [27].
17. Pretend h–1(D) is rgα–closed in P for all closed subset D of Q then it is termed as rgα–continuous [28].
18. Pretend h–1(D) is gr–closed in P for all closed subset D of Q then it is termed as gr–continuous [9].
19. Pretend h–1(D) is rps–closed in P for all closed subset D of Q then it is termed as rps–continuous [25].
20. Pretend h–1(D) is R\*–closed in P for all closed subset D of Q then it is termed as R\*–continuous [15].
21. Pretend h–1(D) is gprw–closed in P for all closed subset D of Q then it is termed as gprw–continuous [16].
22. Pretend h–1(D) is wgrα–closed in P for all closed subset D of Q then it is termed as wgrα–continuous [15].
23. Pretend h–1(D) is swg–closed in P for all closed subset D of Q then it is termed as swg–continuous [23].
24. Pretend h–1(D) is rw–closed in P for all closed subset D of Q then it is termed as rω–continuous [8].
25. Pretend h–1(D) is rgw–closed in P for all closed subset D of Q then it is termed as rgw–continuous [22].

**1.3.2 Definition:** In a map h: (P, τ) →(Q, σ)

1. Pretend h–1(D) is α-closed in P for all α-closed subset D of Q then it is termed as α–irresolute [14].
2. Pretend h–1(D) is semi- closed in P for all semi-closed subset D of Q then it is termed as irresolute [7].
3. Pretend h–1(D) is ω-open in P for all w-closed subset D of Q then it is termed as contra w–irresolute [27].
4. Pretend h–1(D) is semi-open in P for all semi-closed subset D of Q then it is termed as contra irresolute [14].
5. Pretend h–1(D) is regular-open in P for all regular-closed subset D of Q then it is termed as contra r–irresolute [3]
6. Pretend h–1(D) is open in P for all closed subset D of Q then it is termed as contra continuous [11] .
7. Pretend h(D) is rw-open (resp rw-closed) in Q for all rw-open (resp rw-closed) subset D of P then it is termed as rw\*-open (resp rw\*-closed) [8] map.

**Definition 1.3.8:** A topological space (P, τ) is termed as

1. T½ space [23] if each semi-closed set is closed.
2. Tws space [7]if each ws-closed set is closed.

1.4 Closed and open maps and homeomorphism:

Malghan [57] explored and deliberated generalised closed maps. Sundaram [89], Arockirani[3],Nagaveni[65], Sheik John[83], Benchalli [ ] and R. S. Walli [ ], Pushpalatha [79],Crossely and Hildebrand[20],mashhour[60] and Gnanambal[40] has explored generalised open maps, rg-closed maps, wg closed and open maps, w-closed and open maps ,rw- closed and open maps, αrw-closed and open maps, g\*-closed and open maps, pre semi open maps, α-open maps, gpr-closed maps respectively.

The concept of generalised homeomorphism explored and deliberated by Balachandran, Nagaveni [25] Sheik John[30],Vadivel et al, Thangavel [38], Maki has explored rwg- homeomorphism, w-homeomorphism, rgα-homeomorphism, gs-homeomorphism and sg- homeomorphism.

**Definition 1.4.1** A map h: (P, τ) → (Q σ) is said to be

1. α– closed map [15] if h(N) is α–closed in Q ∀closed subset V of P.
2. gspr–closed map [28] if h(N) is gspr–closed in Q ∀closed subset N of P
3. semi–closed map [19] if h(N) is semi–closed in Q ∀closed subset N of P
4. ω–closed map [31] if h(N) is ω–closed in Q ∀closed subset N of P
5. rgα–closed map [33] if h(N) is rgα–closed in Q ∀closed subset N of P
6. gr–closed map [10] if h(N) is gr–closed in Q ∀closed subset N of P
7. g\*p–closed map [22] if h(N) is g\*p–closed in Q ∀closed subset N of P
8. rps–closed map [29] if h(N) is rps–closed in Q ∀closed subset N of P
9. R\*–closed map [14] if h(N) is R\*–closed in Q ∀ closed subset N of P
10. gprw– closed map [17] if h(N) is gprw–closed in Q ∀ closed subset N of P.
11. wgrα– closed map [16] if h(N) is wgrα–closed in Q ∀ closed subset N of P.
12. αg– closed map [21] if h(N) is αg–closed in Q ∀ closed subset N of P.
13. swg– closed map [26] if h(N) is swg–closed in Q ∀ closed subset N of P.
14. rω– closed map [28] if h(N) is rw–closed in Q ∀ closed subset N of P.
15. rgw–closed map [25] if h(N) is rgw–closed in Q ∀ closed subset N of P.
16. regular closed map[30] if h(N) is closed in Q ∀ regular closed set N of P
17. Contra closed map [4] if h(N) is closed in Q ∀ open set N of P.
18. Contra regular closed map [30] if h(N) is r-closed in Q ∀open set N of P.
19. Contra semi-closed map [27] if h(N) is s-closed in Q ∀open set N of P.
20. wg–closed map [26] if h(N) is wg–closed in Q ∀ closed subset N of P.
21. rwg–closed map [26] if h(N) is rwg–closed in Q ∀ closed subset N of P.
22. gs–closed map [3] if h(N) is gs–closed in Q ∀ closed subset N of P.
23. gp–closed map [21] if h(N) is gp–closed in Q ∀ closed subset N of P.
24. gpr–closed map [13] if h(N) is gpr–closed in Q ∀ closed subset N of P.
25. αgr–closed map [34] if h(N) is αgr–closed in Q ∀ closed subset N of P.
26. ωα–closed map [9] if h(N) is ωα–closed in Q ∀ closed subset N of P.

**Definition 1.4.2** A map h: (P, τ) → (Q σ) is said to be

1. semi-open [15] if h(N) is g-open in Q ∀open set N of (P, τ),
2. gpr-open [13] if h(N) is gpr-open in Q ∀open set N of (P, τ),
3. Regular open [29] if h(N) is open in (Q ,σ) ∀ regular open set N of (P, τ).
4. rwg-open [26] if h(N) is rwg-open in Q ∀ open set N of (P, τ),
5. wg-open [26] if h(N) is wg-open in Q ∀ open set N of (P, τ),
6. w-open [31] if h(N) is w-open in Q ∀ open set N of (P, τ).

**Definition 1.4.3** : Map h: P → Qis said to be

1. homeomorphism if h is both open and continuous
2. gspr-homeomorphism if h is both gspr -continuous and gspr –open.
3. gsp - homeomorphism if h is both gsp -continuous and gsp -open.
4. rgb-homeomorphism if h is both rgb -open and rgb –continuous.

**1.5 Locally closed sets and LC-continuous maps.**

Ganster and Reilly [37], Sundaram [89], Arockkiarani and Balachandran ([4], [5]), Park ([74], [76]) and Sheik John [83] have respectively introduced and studied locally, generalized locally, regular generalized locally and w-locally closed sets.

**1.5.1 Definition:** In asubset D of topological space (P, τ)

1. If D=M ∩ N, where M is open and N is closed in (X,τ) then it is called as locally closed (briefly lc) set [5]
2. If D=M∩N, where M is α–open and N is α–closed in (X, τ). then it is called as α-locally closed (briefly αlc) set [2]
3. If D=M∩N, where M is w–open and N is w–closed in (X, τ) then it is called as rw- locally closed (briefly wlc) set [3].
4. If D=M∩N, where M is open and N is semi–closed in (X, τ) then it is called as semi - locally closed (briefly lsc) set [3].

**1.5.2 Definition:** In a map h: (P, τ) → (Q, σ)

1. Pretend h-1(D) is locally closed set in (P, τ) for each open set P of (Q, σ) then it is called as LC-continuous [5].
2. Pretend h-1(D) is α- locally closed set in (P, τ) for each open set D of (Q, σ) then it is called as LC-continuous [2].
3. Pretend h -1(D) is α- locally closed set in (P,τ) for each open set D of (Q, σ) then it is called as rw-LC continuous [3].

**1.5.4 Definition:** A map h :(P, τ)→(Q, σ) is termed

(i) LC-irresolute [37] if h−1(N) is a lc set in (P, τ) ∀ lc-set N in (Q, σ),

(ii) w-LC-irresolute [83] if h−1(N) is a w-lc set in (P, τ) ∀ w-lc set N in (Q, σ),

(iii) GLC-irresolute [12] if h−1(N) is a glc set in (P, τ) ∀ glc set N in (Q, σ),

**1.6 Bitopological spaces.**

Kelly [45] initiated a systematic study of the concept of bitopological spaces in 1963. Furthers various authors, like Arya and Nour [7], Di Maio and Noiri [24], Fukutake [34], Reilly [81], Popa [78], Maki [55], Arockiarani [3], Gnanambal [40] and Sheik Jhon [83] have changed their attention to put the individual concepts of topology to bitopological spaces . Here we define some of the definitions, which are used in our study.

**1.6.1 Definition:** Let i, j∈{1, 2} be fixed integers. In a bitopological space (P, τ1, τ2), a subset D of (P, τ1, τ2) is said to

(i) (i, j)-g#-closed [34] if τj- cl(D)⊂M when D⊂M and M∈τi,

(ii) (i, j)-\*g-closed [3] if τj- cl(D)⊂M when D⊂M and M is regular open in τi,

(iii) (i, j)-g\*-closed [40] if τj- pcl(D) ⊂ M when D ⊂ M and M is regular open in τi,

(iv) (i, j)-gp-closed [35] if τj- cl(τi-int(D)) ⊂ M when D ⊂ M and M∈τi,

(v) (i, j)--closed [83] if τj- cl(D) ⊂ M when D ⊂ M and M is semiopen in τi,

(vi) (i, j)-rb-closed [46] if τj- pcl(D) ⊂ M when D ⊂ M and M∈τi,

(vii) (i, j)-gspr-closed [85] if τj- cl(D) ⊂ M when D ⊂ M and M∈GO(P, τi).

(vii) (i, j)-gsp-closed [85] if τj- cl(D) ⊂ M when D ⊂ M and M∈GO(P, τi).

(vii) (i, j)-rgb-closed [85] if τj- cl(D) ⊂ M when D ⊂ M and M∈GO(P, τi).

The complements of the above mentioned closed sets are their respective open sets.

**1.6.2 Definition:** A map h: (P, τ1, τ2)→(Q σ1, σ2) is termed bi-continuous [55] if h is τ1-σ1-continuous and τ2-σ2-continuous.

**1.6.3 Definition:** A map h: (P, τ1, τ2)→(Q σ1, σ2) is termed strongly-bi-continuous [55] (briefly s-bi-continuous) if h is bi-continuous, τ1-σ2-continuous and τ2-σ1-continuous,

**1.7 Some separation axioms in topological spaces:**

From the literature survey on separation axioms we observed that there is a significant work for different relatively weak form of separation axioms like Tk spaces (k=0, 1, 2), normal and regular axioms in particular several other neighbouring forms of them have been studied in many papers.

Maheshwari and Prasad [64] established and deliberated the new class of space called s-normal space using semi-open sets. It was further studied by Noiri [85], Dorsett [34], and Arya [8], Manshi [79] introduced g-regular and g-normal spaces using g-closed sets. Noiri and Popa [90] further investigated the concepts introduced by Manshi. Sheik John[104] introduced and studied the w-normal ,w-regular using w-closed sets.

**CHAPTER -2**

**ws- CLOSED SETS AND ws-OPEN SETS IN TOPOLOGICAL SPACES**

N. Levine [50] elaborated generalised closed sets in general topology as a generalisation of closed sets. This Knowledge of generalised closed sets was enhanced the improvisation of many new results in general topology. Many topologist namely Balachandran et al [13], Bhattacharya at al [15], Arockirani[3], Gnanambal[39], Malghan [57],Devi[23], Benchalli, R.S. Walli has contributed their works on generalised closed sets, their generalisation and related concept in general topology.

Second section describes a new class of closed sets known as weekly semi closed sets (ws-closed) in topological spaces which are explored and examined. During practise of study few properties of ws-closed are obtained and it was found that every semi- closed is ws-closed sets,

Third Section describes about a new class of closed sets known as weekly semi open sets (ws-open) in topological spaces which are elaborated and examined. During practise of study few properties ws- open sets are obtained. We introduce ws- neighbourhood in topological spaces by using notations of ws-open sets. The notions of ws –interior, ws -closure are explored and studied with few of its basic properties and its basic results are obtained.

**2.2 ws-closed sets and their basic properties.**

In this section, a new class of sets, called weakly semi -closed (briefly, ws-closed) sets in topological spaces are introduced and studied. During this process some of their properties are obtained.

**Definition 2.2.1:** A subset *D* of a space (P,τ) is called ws-closed set if scl(D) ⊆ U , whenever A ⊆ U and U is w-open in (P, τ). We denote the collection of all ws-closed sets in P by WSC(P).

First we prove that the class ws-closed sets properly lies between the class all semi-closed sets and generalised semi pre-closed sets in topological spaces.

**Theorem 2.2.2:**- Each semi-closed set in P is ws-closed set but inverse is untrue

**Proof**: Let D be semi-closed set in P. Let M be any w-open set in P, s.t D ⊆ M. Since D is semi-closed, we have scl(D) = D ⊆ M, we have scl(D) ⊆ M. Hence D is ws-closed set in P.

We use example 2.2.3 to prove the inverse of theorem is untrue.

**Example 2.2.3:** Let P= {1, 2, 3, 4} and = {P, ϕ, {1}, {2},{1, 2}, {1, 2, 3}} then the set D= {1,2,3} is ws-closed set but not semi-closed in P.

**Theorem 2.2.4:**

1. If D -C(P) is ws-C(P) but inverse is untrue.
2. If D g#-C(P) is ws-C(P) but inverse is untrue.
3. If D \*g-C(P) is ws-C(P) but inverse is untrue.
4. If D g#s-C(P) is ws-C(P) but inverse is untrue.
5. If D g\*-C(P) is ws-C(P) but inverse is untrue.
6. If D gp-C(P) is ws-C(P) but inverse is untrue.
7. If D -C(P) is ws-C(P) but inverse is untrue.
8. If D regular-C(P) is ws-C(P) but inverse is untrue
9. If D rb-C(P) is ws-C(P) but inverse is untrue.

Proof: Let D be α- (respectively, g#, \*g, g#s, g\*,gp,, regular and rb) closed set in P. Let M be any w-open set in P, s.t D ⊆ M. Since D is α- (respectively, g#, \*g, g#s, g\*,gp,, regular and rb) closed in P, we have scl(D) = D ⊆ M, we have scl(D) ⊆ M. Hence D is ws-closed set in P.

**Example 2.2.5 :** Let P= {1, 2, 3, 4} and ={P, ϕ, {1}, {2},{1, 2}, {1, 2, 3}} then the set D= {1} is ws-closed set but not α- (respectively, g#, \*g, g#s, g\*,gp,, regular and rb) closed in P.

**Theorem 2.2.6:** Each ψ -closed set in P is ws- closed set but inverse is untrue

**Proof**: Let A be ψ -closed set in P. Let M be any w-open set in P, s.t D ⊆ M. Since D is ψ -closed, we have scl(D) = D ⊆ M, we have scl(D) ⊆ M. Hence D is ws-closed set in P.

We use example 2.2.7 to prove the inverse of theorem is untrue.

**Example 2.2.7:** Let P= {1, 2, 3, 4} and = {P, ϕ, {1}, {2},{1, 2}, {1, 2, 3}} then the set D= {1,2,3} is ws-closed set but not ψ -closed in P.

**Corollary 2.2.8:** If D closed set is ws- C(P) set in P.

**Proof:** Each closed set is α-closed and follows from Theorem 2.2.4

We use example 2.2.9 to prove the inverse of theorem is untrue.

**Example 2.2.9:** Let P= {1, 2, 3, 4} and = {P, ϕ, {1}, {2}, {1, 2}, {1, 2, 3}} then the set D = {1} is ws-closed set but not closed in P.

**Theorem 2.2.10:** If D ws- C(P) is gspr-closed set in P but inverse is untrue

**Proof:** Take up D ws- C(P), let M be any regular open set in P s.t D M. since each regular open set is w- open set and D is ws-closed set ,we have scl(D)M,

Therefore scl(D)M ,M is regular open in P, D is gspr –C(P).

We use example 2.2.11 to prove the inverse of theorem is untrue.

**Example 2.2.11:** Let P = {1, 2, 3 4}, τ = **{**P,φ,{1, 2}, {3, 4}**}.** Then the set D= {1} is gspr -closed set but not ws-closed set in P.

**Corollary 2.2.12:**

i) If D ws-C(P) then D gsp- C(P) but inverse is untrue.

ii) If D ws-C(P) D rgb-C(P) but inverse is untrue.

Proof:

1. Follow from [2010] Govindappa Navalagi, S.V. Gurushantanavar, and Chandrashekarappa A.S.

Each gspr-C(P) set is gsp-C(P) set and then follows from Theorem 3.25

1. Follow from [2013] K. Mariappa, S. Sekar, each rgb-closed is gsp-closedset, then follows from Corollary 3.27(i).

We use example 2.2.13 to prove the inverse of above Corallary is untrue.

**Example 2.2.13:** Let P = {1, 2, 3 4}, τ = **{**P, φ, {1, 2}, {3, 4}**}.** Then the set D= {2} is gsp(rgb) -closed set but not ws-closed set in P.

**Remark 2.2.14:** The following examples shows that ws-C(P) sets are independent of rg-C(P) sets, gpr-C(P) sets, wgrα-C(P) sets, pgrα-C(P) sets, -C(P) sets, α\*\*g-C(P) sets, gprw-C(P) sets, rgw-C(P) sets, rw-C(P) sets, rwg-C(P) sets, rgα-C(P) sets, αgr-C(P) sets, gα\*\*-C(P) sets, βwg\*\*-C(P) sets.

**Example 2.2.15:** Let P = {1, 2, 3 4}, τ = **{**P,φ, {1}, {2}, {1, 2}, {1, 2, 3}**}.** Then

1. Closed sets in (P, τ) are P, φ,{4},{3, 4},{1, 3, 4}, {2, 3, 4}.
2. ws-Closed in(P, τ) are P, φ, {1},{2},{3},{4},{1, 3},{1, 4},{2, 3},{2, 4},{3, 4}, {1, 2, 4}, {1, 3, 4},{2, 3, 4}.
3. rg-Closed in(P, τ) are P, φ,{3},{4},{1, 2},{1, 3},{1 4},{2, 3},{2, 4},{3, 4}, {1, 2, 3}, {1, 2,  3}, {1, 3, 4},{2, 3, 4}.
4. gpr -Closed in(P, τ) are P, φ,{3},{4},{1, 2},{1, 3},{1 4},{2, 3},{2, 4},{3, 4}, {1, 2, 3}, {1, 2,  3}, {1, 3, 4},{2, 3, 4}.
5. wgrα -Closed in (P, τ) are P, φ,{3},{4},{1, 2},{1, 3},{1 4},{2, 3},{2, 4},{3, 4}, {1, 2, 3}, {1, 2,  3}, {1, 3, 4},{2, 3, 4}.
6. pgrα -Closed in(P, τ) are P, φ,{3},{4},{1, 2},{1, 3},{1 4},{2, 3},{2, 4},{3, 4}, {1, 2, 3}, {1, 2,  3}, {1, 3, 4},{2, 3, 4}.
7. -Closed in(P, τ) are P, φ,{3},{4},{1, 2},{1, 3},{1 4},{2, 3},{2, 4},{3, 4}, {1, 2, 3}, {1, 2,  3}, {1, 3, 4},{2, 3, 4}.
8. α\*\*g -Closed in(P, τ) are P, φ,{3},{4},{1, 2},{1, 3},{1 4},{2, 3},{2, 4},{3, 4}, {1, 2, 3}, {1, 2,  3}, {1, 3, 4},{2, 3, 4}.
9. gprw -Closed in(P, τ) are P, φ,{3},{4},{1, 2},{3, 4},{1, 2, 3},{1, 2, 4},{1, 3, 4},{2, 3, 4}.
10. rgw -Closed in(P, τ) are P, φ,{3},{4},{1, 2},{3, 4},{1, 2, 3},{1, 2, 4},{1, 3, 4}, {2, 3, 4}.
11. rw-Closed in(P, τ) are P, φ,{4},{1, 2},{3, 4},{1, 2, 3},{1, 2, 4},{1, 3, 4},{2, 3, 4}.
12. rwg-Closed in(P, τ) are P, φ,{1},{2},{3},{4},{1, 3},{1, 4},{2, 3},{2, 4},{3, 4}, {1, 2, 4}, {1, 3, 4},{2, 3, 4}.
13. rgα-Closed in(P, τ) are P, φ,{1},{2},{3},{4},{1, 3},{1, 4},{2, 3},{2, 4},{3, 4}, {1, 2, 4}, {1, 3, 4},{2, 3, 4}.
14. αgr-Closed in(P, τ) are P, φ,{1},{2},{3},{4},{1, 3},{1, 4},{2, 3},{2, 4},{3, 4}, {1, 2, 4}, {1, 3, 4},{2, 3, 4}.
15. gα\*\*-Closed in(P, τ) are P, φ,{1},{2},{3},{4},{1, 3},{1, 4},{2, 3},{2, 4},{3, 4}, {1, 2, 4}, {1, 3, 4},{2, 3, 4}.
16. βwg\*\*-Closed in(P, τ) are P, φ,{1},{2},{3},{4},{1, 3},{1, 4},{2, 3},{2, 4},{3, 4}, {1, 2, 4}, {1, 3, 4},{2, 3, 4}.

Therefore {1} is ws- closed in P but not rg (resp. gpr, wgrα , pgrα, , α\*\*g, gprw, rgw, rw, rwg, rgα, αgr, gα\*\* βwg\*\*) closed set in P.

**Remark 2.2.16:** The following examples clearly represent that ws-closed sets are independent of sets.\*g, Mildly g , wg, gwα-, g\*p, η, gp, βwg\*, \*\*gα , , , αgs, wα-, #gα-, og#-, g\*-pre and g#p#)closed sets.

**Example 2.2.17:** Let P = {1, 2, 3}, τ1 = { P, φ, {1}, {2},{1, 2}} and τ2 = { P, φ, {1}, {2, 3}}. Then

1. closed sets in (P, τ1) are P, φ, {3},{1, 3},{2, 3}.
2. ws- closed sets (P, τ1) are P, φ,{1}, {2},{3},{1, 3},{2, 3}.
3. \*g- closed sets (P, τ1) are P, φ, {3},{1, 3},{2, 3}.
4. Mildly g- closed sets (P, τ1) are P, φ, {3},{1, 3},{2, 3}.
5. wg- closed sets (P, τ1) are P, φ, {3},{1, 3},{2, 3}.
6. gwα- closed sets (P, τ1) are P, φ, {3},{1, 3},{2, 3}.
7. g\*p- closed sets (P, τ1) are P, φ, {3},{1, 3},{2, 3}.
8. η- closed sets (P, τ1) are P, φ, {3},{1, 3},{2, 3}.
9. gp- closed sets (P, τ1) are P, φ, {3},{1, 3},{2, 3}.
10. βwg\*- closed sets (P, τ1) are P, φ, {3},{1, 3},{2, 3}.
11. \*\*gα- closed sets (P, τ1) are P, φ, {3},{1, 3},{2, 3}.
12. - closed sets (P, τ1) are P, φ, {3},{1, 3},{2, 3}.
13. - closed sets (P, τ1) are P, φ, {3},{1, 3},{2, 3}.}.
14. αgs - closed sets (P, τ1) are P, φ, {3},{1, 3},{2, 3}.
15. #gα- closed sets (P, τ1) are P, φ, {3},{1, 3},{2, 3}.
16. g\*-pre closed sets (P, τ1) are P, φ, {3},{1, 3},{2, 3}.
17. g#p#- closed sets (P, τ1) are P, φ, {3},{1, 3},{2, 3}.
18. gα\*- closed sets ( P, τ1) are P, φ, {3},{1, 3},{2, 3}.
19. αg- closed sets ( P, τ1) are P, φ, {3},{1, 3},{2, 3}.
20. g\*\*- closed sets (P, τ1) are P, φ, {3},{1, 3},{2, 3}. and also
21. closed sets (P, τ2) are P, φ,{1},{2, 3}.
22. ws- closed sets ( P, τ2) are P, φ,{1} ,{2, 3}.
23. \*g- closed sets (P, τ2) are P, φ, {1}, {2},{3},{1, 2},{1, 3},{2, 3}.
24. Mildly g- closed sets (P, τ2) are P, φ, {1}, {2},{3},{1, 2},{1, 3},{2, 3}.
25. wg- closed sets (P, τ2) are P, φ, {1}, {2},{3},{1, 2},{1, 3},{2, 3}.
26. gwα- closed sets ( P, τ2) are P, φ, {1}, {2},{3},{1, 2},{1, 3},{2, 3}.
27. g\*p- closed sets (P, τ2) are P, φ, {1}, {2},{3},{1, 2},{1, 3},{2, 3}.
28. η- closed sets (P, τ2) are P, φ,{1}, {2},{3},{1, 2},{1, 3},{2, 3}.
29. gp- closed sets (P, τ2) are P, φ, {1}, {2},{3},{1, 2},{1, 3},{2, 3}.
30. βwg\*-Closed in(P, τ2) are P, φ, {1}, {2},{3},{1, 2},{1, 3},{2, 3}.
31. \*\*gα- closed sets (P, τ2) are P, φ, {1}, {2},{3},{1, 2},{1, 3},{2, 3}.
32. - closed sets (P, τ2) are P, φ, {1}, {2},{3},{1, 2},{1, 3},{2, 3}.
33. - closed sets (P, τ2) are P, φ, {1}, {2},{3},{1, 2},{1, 3},{2, 3}.
34. αgs - closed sets (P, τ2) are P, φ, {1}, {2},{3},{1, 2},{1, 3},{2, 3}.
35. #gα- closed sets (P, τ2) are P, φ, {1}, {2},{3},{1, 2},{1, 3},{2, 3}.
36. g\*-pre closed sets (P, τ2) are P, φ, {1}, {2},{3},{1, 2},{1, 3},{2, 3}.
37. g#p#- closed sets (P, τ2) are P, φ, {1}, {2},{3},{1, 2},{1, 3},{2, 3}.
38. gα\*- closed sets ( P, τ2) are P, φ, {1}, {2},{3},{1, 2},{1, 3},{2, 3}.
39. αg- closed sets ( P, τ2) are P, φ, {1}, {2},{3},{1, 2},{1, 3},{2, 3}.
40. g\*\*- closed sets ( P, τ2) are P, φ, {1}, {2},{3},{1, 2},{1, 3},{2, 3}.

Therefore {2} is ws-closed set in (P, τ1) but not in \*g (resp. Mildly g, wg, gwα, g\*p, η, gp, βwg\*,\*\*gα , , ,αgs α, #gα, og#, g\*-preC(P) , g#p#, gα\*,αg, g\*\*)C(P) set in (P, τ1).

Meanwhile {2} in \*g-(resp. gwα, g\*p, η, gp, βwg\*, \*\*gα , , , αgs, α, #gα, og#, g\*-pre , g#p#, gα\*, αg, g\*\*)C(P) in (P, τ2) but not ws-C(P) set in (P, τ2).

**Remark 2.2.18:** The following example clearly represents that ws-closed sets are independent of sets g, sg, gs, gα, sgb, rg\*b, pgpr, gαb and rps-closed sets in P

**Example 2.2.19:** Let P = {1, 2, 3, 4}, τ1 ={P, φ, {1}, {1, 2},{1, 2, 3}} and τ2 ={ P, φ, {1, 2},{3, 4}}. Then

1. closed sets in (P, τ1) are P, φ, {4},{3, 4},{2, 3, 4}.
2. ws- closed sets (P, τ1) are P, φ,{1},{2},{3},{4},{1, 3},{1, 4},{2, 3},{2, 4},{3, 4}, {1, 2, 4}, {1, 3, 4},{2, 3, 4}.
3. g- closed sets (P, τ1) are P, φ,{4},{1, 4},{2, 4},{3, 4}{1, 2, 4},{1, 3, 4},{2, 3, 4}.
4. sg- closed sets (P, τ1) are P, φ,{2},{3},{4},{2, 3},{2, 4},{3, 4},{2, 3, 4}.
5. gs- closed sets (P, τ1) are P, φ,{2},{3},{4},{2, 3},{2, 4},{3, 4},{2, 3, 4}.
6. gα- closed sets (P, τ1) are P, φ,{2},{3},{4},{2, 3},{2, 4},{3, 4},{2, 3, 4}.
7. sgb - closed sets (P, τ1) are P, φ,{2},{3},{4},{2, 3},{2, 4},{3, 4},{2, 3, 4}.
8. rg\*b- closed sets (P, τ1) are P, φ,{2},{3},{4},{2, 3},{2, 4},{3, 4},{2, 3, 4}.
9. pgpr- closed sets (P, τ1) are P, φ,{2},{3},{4},{2, 3},{2, 4},{3, 4},{2, 3, 4}.
10. gαb- closed sets (P, τ1) are P, φ,{2},{3},{4},{2, 3},{2, 4},{3, 4},{2, 3, 4}.
11. rps- closed sets (P, τ1) are P, φ,{2},{3},{4},{2, 3},{2, 4},{3, 4},{2, 3, 4}. and also
12. Closed sets (P, τ2) are P, φ,{3, 4},{1, 2}.
13. ws- closed sets (P, τ2) are P, φ, {1, 2},{3, 4}.
14. g- closed sets (P, τ2) are P, φ,{1},{2},{3},{4},{1, 2},{, 3},{, 4},{2, 3},{2, 4}, {3, 4},{1, 2, 3},{1, 2, 4},{1, 3, 4},{2, 3, 4}.
15. sg- closed sets (P, τ2) are P, φ,{1},{2},{3},{4},{1, 2},{, 3},{, 4},{2, 3},{2, 4}, {3, 4},{1, 2, 3},{1, 2, 4},{1, 3, 4},{2, 3, 4}.
16. gs- closed sets (P, τ2) are P, φ,{1},{2},{3},{4},{1, 2},{, 3},{, 4},{2, 3},{2, 4}, {3, 4},{1, 2, 3},{1, 2, 4},{1, 3, 4},{2, 3, 4}.
17. gα- closed sets (P, τ2) are P, φ,{1},{2},{3},{4},{1, 2},{, 3},{, 4},{2, 3},{2, 4}, {3, 4},{1, 2, 3},{1, 2, 4},{1, 3, 4},{2, 3, 4}.
18. sgb - closed sets (P, τ2) are P, φ,{1},{2},{3},{4},{1, 2},{, 3},{, 4},{2, 3},{2, 4}, {3, 4},{1, 2, 3},{1, 2, 4},{1, 3, 4},{2, 3, 4}.
19. rg\*b- closed sets (P, τ2) are P, φ,{1},{2},{3},{4},{1, 2},{, 3},{, 4},{2, 3},{2, 4}, {3, 4},{1, 2, 3},{1, 2, 4},{1, 3, 4},{2, 3, 4}.
20. pgpr- closed sets (P, τ2) are P, φ,{1},{2},{3},{4},{1, 2},{, 3},{, 4},{2, 3},{2, 4}, {3, 4},{1, 2, 3},{1, 2, 4},{1, 3, 4},{2, 3, 4}.
21. gαb- closed sets (P, τ2) are P, φ,{1},{2},{3},{4},{1, 2},{, 3},{, 4},{2, 3},{2, 4}, {3, 4},{1, 2, 3},{1, 2, 4},{1, 3, 4},{2, 3, 4}.
22. rps- closed sets (P, τ2) are P, φ,{1},{2},{3},{4},{1, 2},{, 3},{, 4},{2, 3},{2, 4}, {3, 4},{1, 2, 3},{1, 2, 4},{1, 3, 4},{2, 3, 4}.

Therefore {1} is ws-C(P) set in (P, τ1) but not g-C(P) (resp. sg-C(P), gs-C(P), gα-C(P), sgb-C(P) sets, rg\*b-C(P), pgpr-C(P), gαb-C(P), rps-C(P)) set in (P, τ1).

Meanwhile {1} is g-(resp. sg-, gs, gα, sgb, rg\*b, pgpr, gαb ,rps) closed set in (P, τ2) but not ws-closed set in (P, τ2).

**Remark 2.2.20:** The next example represents that ws-closed will be independent of R\*, rgβ pgrα, rgw and gprw-close sets in P.

**Example 2.2.21:** Let P = {1, 2, 3}, and τ = {P,φ, {1}, {2}, {1, 2}}. Then

1. closed sets (P, τ) are P, φ,{3},{1, 3},{2, 3}.
2. ws- closed sets (P, τ) are P, φ, {1},{2},{3},{2, 3}, {1, 3}.
3. R\* - closed sets (P, τ) are P, φ,{3},{1, 2},{2, 3},{1, 3}.
4. rgβ - closed sets (P, τ) are P, φ,{3},{1, 2},{2, 3},{1, 3}.
5. pgrα- closed sets (P, τ) are P, φ,{3},{1, 2},{2, 3},{1, 3}.
6. w- closed sets (P, τ) are P, φ,{3},{1, 2},{2, 3},{1, 3}.
7. gprw - closed sets (P, τ) are P, φ,{3},{1, 2},{2, 3},{1, 3}.

Therefore {1} is ws-C(P) set in P but not R\*(respectively. rgβ , pgrα , rgw , gprw)closed set in P.

**Remark 2.2.22**: From the results discussed above and the known facts, we have the following implications.

Regular closed

Closed

g\*-closed

α-closed

\*gα-closed

**Semi closed**

g#s-closed

**gspr-closed**

rb-closed

**gsp-closed**

gξ\*-closed

-closed

rgb-closed

Ψ-closed

g#-closed

rg-closed, gpr-closed, wgrα-closed, pgrα-closed, -closed, α\*\*g-closed, gprw-closed, rgw-closed, rw-closed, rwg-closed, rgα-closed, αgr-closed, gα\*\*-closed, βwg\*\*-closed, \*g-closed, Mildly g-closed , wg-closed , gwα-closed , g\*p-closed , η-closed , gp-closed , βwg\*-closed ,\*\*gα-closed , -closed , -closed ,αgs-closed α-closed , #gα-closed , og#-closed , g\*-preclosed , g#p#-closed , gα\*-closed ,αg-closed, g\*\*-closed, g-closed, sg-closed, gs-closed, gα-closed, sgb-closed sets, rg\*b-closed, pgpr-closed,rps-closed

A B means the set A implies the set B but inverse is untrue

A B means the set A and the set B are independent of each other

Note: A B Means A implies B, but inverse is untrue.

A B Means A and B are independent.

**Theorem 2.2.23:** The union of two ws-closed subsets of P is ws-closed.

**Example 2.2.24:** Let P = {1, 2, 3} and τ = {φ, P, {1}, {2}, {1, 2}} thus the sets A= {1} and B= {2} are ws- C(P) set but AB = {1,2 } is not a ws-C(P).

**Theorem 2.2.25:** The intersection of two ws-closed subsets of P is ws-closed.

**Proof:** Take up D and E ws- C(P). Let U be any semiopen set P : (D∩E)⊆U that is D⊆U and E⊆U. Since D and E are ws-C(P) then scl(D) ⊆ U and scl(E)⊆U and we know that (scl(D)scl(E)) = scl(DE)⊆U. Therefore DE is ws-closed in P.

**Theorem 2.2.26:** Assuming subset D of a topological space P is ws- C(P) then scl(D)-D any non-empty open set in P but inverse is untrue.

**Proof:** Take up D ws-C(P) and Pretend F be an non empty w-C(P) subset of scl(D)-D . F scl(D)-D Fscl(D)(P-D) Fscl(D) ---(1) & FP-D

DP-F and P-F is w-open set and D is an ws- C(P), scl(D)P-F

FP-scl(D) ---(2) from equations (1) and (2) we get Fscl(D)(P-scl(D))=

F= thus scl(D)-D will not contain any non-empty w-C(P) .

We use example 2.2.27 to prove the inverse of theorem is untrue.

**Example 2.2.27:** Take P={1,2,3,4} τ=**{** P,φ,{1},{1, 2},{1,2,3}**}** then the set D={2} scl{2}={2},scl{D}-D={2} any non-empty w-C(P) but D is not ws-C(P).

**Theorem 2.2.28:** If D ws-C(P) and D⊆E⊆scl(D) then E is also ws-C(P)**.**

**Proof:** Take up D ws-C(P) , : E⊆scl(D). Assume U be a w-open set of P : E⊆U then D⊆U. Since D is ws-C(P) we have scl(D)⊆U and D⊆U.Now E⊆ scl(D) scl(E)⊆scl(scl(D))=scl(D)⊆U. That is scl (E)⊆U. Therefore E is a ws-C(P).

We use example 2.2.29 to prove the inverse of theorem is untrue.

**Example 2.2.29:** Take up P = {1,2,3 }, τ = {φ, P, {1}, {2}, {1, 2}} then the set D={1}, E={1,3} : D and E are ws-C(P) but D⊆E scl(D) because scl(D)={1}.

**Theorem 2.2.30:** Pretend (P,τ) is a topological space then ∀ x belongs to P the set P-{x} is ws-C(P) or semi open.

**Proof:** Let xP. Suppose P-{x} semiopen set. Then P is the only semiopen set containing P-{x} that is P-{x} ⊆ P cl(P-{x}) ⊆cl(P) cl(P-{x})⊆P. Therefore P-{x} is ws-C(P) .

**Theorem 2.2.31:** Take up D ws-C(P) . Then D is semi-closed iff scl(D)-D is w-open**.**

**Proof:** Necessity: Pretend D is g-C(P) . Then scl(D)=D that is scl(D)-D=φ which is w-open set in P.

Sufficiency: Pretend D is ws-C(P) and scl(D)-D is w-open, By the Theorem **2.2**.**26** scl(D)-D=φ scl(D)=D. Therefore D is semi-C(P).

**Theorem 2.2.32:** Let D⊆Q⊆P and Pretend that D is ws-C(P). Then D is ws-closed relative to Y.

**Proof:** Let D⊆Q∩G where G is w-open. Since D is ws-C(P), then D⊆G and scl(D)⊆G. This implies that Q ∩ scl(D) ⊆Q∩G where Q ∩ scl(D)is closed set of D in Q. Thus D is ws-closed relative to Q

**Theorem 2.2.33:** In a topological space P if SO(P) ={P, φ} then each subset P is a ws-C(P)**.**

**Proof:** Take up P be a topological space and SO(P)={P, φ}. Let D be any subset of P. Pretend D=φ. Then φ is ws-C(P). Pretend D≠φ. Then P is the only semiopen set containing A and so scl(D)⊆P. Hence D is ws-C(P).

We use example 2.2.34 to prove the inverse of theorem is untrue.

**Example 2.2.34**: Let P = {1, 2, 3}, τ = {φ, P, {1}, {2, 3}}. Then each subset of (P,τ) is a ws-C(P), but SO= {φ, P, {1}, {2, 3}}.

**Theorem 2.2.35:** If D regular open and gspr- closed then D ws-C(P).

**Proof:** Take up D regular open and gspr-closed in P. Let M be any w-open set in P : D⊆M. Since D is regular open and gspr-closed, by definition, scl(D)⊆D then scl(D)⊆D⊆U. Hence D is ws-C(P) .

**Theorem 2.2.36:** If D regular open and rgb-closed then D ws-C(P).

**Proof:** Take up D be regular open and rgb-closed in P. Let M be any w-open set in P. : D⊆M. Seeing that D is regular open and rgb-closed in P, by definition, scl(D)⊆D thus scl(D)⊆D⊆M. Hence D is ws-C(P) .

**Theorem 2.2.37:** If D semiopen and swg\*-closed then D ws-C(P).

**Proof:** Let D semiopen and swg\*- closed in P. Let M be any w- open set in P : D⊆M. Since D is semiopen and swg\*-closed in P, by definition, scl(D)⊆D thus scl(D)⊆D⊆M. Hence D is ws- C(P).

**Theorem 2.2.38:** If D is semiopen and swg-closed then D ws-C(P).

**Proof:** Take up D be semiopen and swg-closed in P. Let M be any w- open set in P : D⊆M. Since D is semiopen and swg-closed in P, by definition, scl(D)⊆D implies scl(D)⊆D⊆M. Hence D is ws-C(P) .

**Theorem 2.2.39:** If D is semiopen and sg-closed then D ws-C(P).

**Proof:** Take up D be semiopen and sg-closed in P. Let M be any w- open set in P : D⊆M. Since D is semiopen and sg-closed in P, by definition, scl(D)⊆D implies scl(D)⊆D⊆M. Hence D is ws-C(P).

**Theorem 2.2.40:**  If Dsemiopen and sgb-closed then D ws- C(P).

**Proof:** Take up D be semiopen and sgb-closed in P. Let M be any w- open set in P : D⊆M. Since D is semiopen and sgb-closed in P, by definition, scl(D)⊆D implies scl(D)⊆D⊆M. Hence D is ws-C(P).

**Theorem 2.2.41:** If D semiopen and αgs-closed then D ws-C(P).

**Proof:** Take up D semiopen and **αgs**-closed in P. Let M be any w- open set in P : D⊆M. Since D is semiopen and αgs -closed in P, by definition, scl(D)⊆D implies scl(D)⊆D⊆M. Hence D is ws-C(P).

**Theorem 2.2.42:** If D β-open and βwg\*-closed then D ws-C(P).

**Proof:** Let D β-open and βwg\*-closed in P. Let M be any regular semiopen set in P : D⊆M. Since D is β-open and βwg\*-closed in P, by definition, gcl(D)⊆D implies gcl(D)⊆D⊆M. Hence D is ws-closed in P.

**Theorem 2.2.43:** If D both open and g-closed then D ws-C(P).

**Proof:** Let D open and g-closed in P. Let M be any regular open set in P D⊆M. by definition, cl(D)⊆D⊆M and gcl(D)=D. This implies cl(D)⊆gcl(D) ⊆D⊆M gcl(D)⊆M. Hence D is ws-C(P).

**Theorem 2.2.44:** If D regular semiopen and rw-closed then D ws-C(P).

**Proof:** Take up D regular semiopen and rw-C(P) in P. Let M be any w-open set in P D⊆M. Now D⊆D by hypothesis cl(D)⊆D then we know that cl(D) ⊆scl(D)⊆D. Hence scl(D)⊆M therefore D is ws-C(P) .

**Theorem 2.2.45:** If D regular semiopen and R\*-closed then D ws-C(P).

**Proof:** Take up D regular semiopen and R\*-closed in P. Let M be any w-open set in P D⊆M. Now D⊆D by hypothesis cl(D)⊆D then we know that cl(D) ⊆scl(D)⊆D. Hence scl(D)⊆M therefore D is ws-C(P).

**Theorem 2.2.46:** If D regular semiopen and gprw-closed then D ws-C(P).

**Proof:**Take up D be regular semiopen and gprw –closed in P. Let M be any w-open set in P : D⊆M. Now D⊆D by hypothesis cl(D)⊆D then we know that cl(D) ⊆scl(D)⊆D. Hence scl(D)⊆M therefore D is ws-C(P).

**Theorem 2.2.47:** If D is regular semiopen and rgw-closed then D ws-C(P).

**Proof:** Take up D be regular semiopen and rgw –closed in P. Let M be any w-open set in P : D⊆M. Now D⊆D by hypothesis cl(D)⊆D then we know that cl(D) ⊆scl(D)⊆D. Hence scl(D)⊆M therefore D is ws-C(P) .

**2.3 ws-open sets in topological spaces:**

**Definition 2.3.1:** A subset D of a topological space (P,τ) is called a ws-open set in P if Dc is a ws-closed.

**Theorem 2.3.2:** For any topological space (P,τ) we have the following

1. Each regular (respectively, semi, α, g#, \*gα, g#s, rb, , gξ\*, αgp, ψ) open set in P is ws-open in P.
2. Each ws-open set in P is gspr (respectively gsp, rgb) open set in P.

**Theorem 2.3.3:** If D and E are ws-open sets in space P then DE is also a ws-open set in P.

**Proof:** Take up D and E be ws-open sets in P. Then Dc and Ec are ws-C(P), by Theorem 2.2.27 DcEc is also ws-C(P). That is DcEc = (AE)c is ws-C(P) . Therefore (AE) is ws-open set in P.

**Remark 2.3.4:** Intersection of ws-open sets in P is not a ws-open set in P.

**Example 2.3.5:** Take up P = {1,2, 3}, τ = {φ, P, {1}, {2}, {1, 2}}. ws- open = {φ, P, {1}, {2},   
{1, 2},{1,3},{2,3}}. If D= {1,3} and E={2,3} then D and E are ws-open sets but AE={3}is not a ws-open set in P.

**Theorem 2.3.6:** A subset D of a topological space P is ws-open iff M⊆ wsint (D) whenever M⊆D and M is w-open in P.

**Proof:** Pretend D is ws-open. M⊆D and M is w-open. Then Dc⊆Mc and Mc is also w-open. By the definition of ws-C(P) scl(Dc)⊆Mc . But scl(Dc)=(wsint(D))c.=P-wsint(D) This implies M⊆wsint(D)

Inversely,

Suppose that M⊆ wsint(D) whenever M⊆D , M is w-open in P. Let Dc⊆F where F is w-open in P. Fc⊆D and Fc is w-open in P. we know that Fc⊆wsint(D) and (wsint(D))c⊆F. Since scl(Dc)= (wsint(D))c we have scl(Dc)⊆M. Thus Dc is ws-C(P). That is D is ws-open set.

**Theorem 2.3.7:** If sint(D)⊆E⊆D and D is ws-open set then E is ws-open set.

**Proof:** Let sint(D)⊆E⊆D. Thus P-D⊆P-E⊆P-sint(D). That is P-D⊆P-E⊆scl(P-D) since P-D is ws-C(P) by theorem 2.3.8 P-E is ws-C(P). Therefore E is ws-open set.

**2.4 ws-closure in topological spaces**

**Definition 2.4.1:** For a subset D of (P, τ), ws-closure of D is denoted by wscl(D) and it is defined as wscl(D)=∩{G: D⊆G, G⊆wsC(P)} or intersection of all ws-C(P)s containing D.

**Theorem 2.4.2:** If D and E are subsets of space (P, τ) then

1. wscl(P)=P, wscl(φ)=φ
2. D⊆wscl(D)
3. if E is any ws-C(P) containing D then wscl(D)⊆E
4. If D⊆E then wscl(D)⊆wscl(E)
5. wscl(D)=wscl(wscl(D))
6. wscl(DE)=(wscl(D)wscl(E))

**Proof:**

i) From ws-closure defination, wscl(P) =Intersection of all ws-C(P) containing P =P ∩ws-C(P) containing P=P∩P =P, Therefore wscl(P)=P. By the definition of ws-closure, wscl(φ)=intersection of all ws-C(P)s containing φ =φ∩ any ws-C(P) containing φ= φ∩φ=φ. Therefore wscl(φ)=φ.

ii) By the definition of ws-closure of D, it is obvious that D⊆ wscl(D).

iii) Let E be any ws-C(P) containing D. Since wscl(D) is the intersection of all ws-C(P) containing D. wscl(D) is contained in each ws-C(P) containing D. Hence in particular wscl (D)⊆E.

iv) Let D and E be subsets of (P,τ) ∋: D⊆E. By the definition of ws-closure, wscl(E)=∩{F: E⊆Fwscl(P)}. If E⊆Fwscl(P) then wscl(E)⊆F. Since D⊆E ⊆Fwscl(P), we have wscl(D)⊆F, wscl(D)⊆∩{ F: E⊆Fwscl(P)}=wscl(E). Therefore wscl(D)⊆wscl(E).

v) Let D be any subset of P. By the definition of ws-closure, wscl(D)=∩{F: A⊆F wscl(P)}.

vi) If D⊆FwsC(P) then wscl(D)⊆F. Since F is a ws-C(P) containing wscl(D), by iii) wscl(wscl(D))⊆F. Hence wscl(wscl(D))⊆∩{F: D⊆Fwscl(P)}=wsC(D). That is wscl(wscl(D))=wscl(D).

Let D and E be subsets of (P,τ). Clearly D⊆(DE) and E⊆(DE). From iv) (wscl(D)wscl(E))⊆wscl(DE) …(1) Now we need to determine that wscl(DE) ⊆wscl(D)wscl(E). Suppose P**(**wscl(D)wscl(E)) then their exists ws-closed sets D1and E1 ∋: D⊆D1, E⊆E1 and P**(**D1E1) Thus we have (DE)⊆(D1E1)and (D1E1) is a ws-C(P) by Theorem … ∋: P(D1E1) . Thus Pwscl(DE). Hence wscl(DE)⊆(wscl(D)wscl(E)) ….(2). From (1) and (2) we have wscl(DE)=(wscl(D)wscl(E)).

**Definition 2.4.3:** For a subset D of (P,τ), ws-interior of D is denoted by wsint(D) and it is defined as wsint(D)={G: G⊆D and G is ws-open in P} or {G: G⊆D and G wsO(P)} or wsint(D) is the union of all ws-open sets contained in D.

**Theorem2.4.4:** Let D and E be subsets of space P then

i) wsint (P)=P, wsint(φ)=φ

1. wsint(D)⊆D
2. if E is any ws-open set contained in D then E⊆wsint(D)
3. If D⊆E then wsint(D)⊆wsint(E)
4. wsint(D)=wsint(wsint(D))
5. wsint(D∩E)=(wsint(D)∩wsint(E)).

Proof:

i) and ii) follows by the definition of ws-interior of D.

iii) Let E is any ws-open set E⊆D. Let PE, E be ws-open set contained in D, P is an interior point of D that is Pwsint(D). Hence E⊆ wsint(D).

similarly Proof of iv), v) and vi) can be proved.

**Theorem 2.4.5: Pretend that a subset D of P is ws-open then wsint(D)=D**

**Proof:** Take up D be ws-open set of P. we know that wsint(D)⊆D….(1) Also D is ws-open set contained in D from Theorem 2.3.7 iii) D⊆wsint(D)….(2). Hence from (1) and (2) wsint(D) =D.

**Theorem 2.4.6: If D and E are subsets of space P then (wsint(D)wsint(E))⊆wsint(DE)**

**Proof:** As we know D⊆(DE) and E⊆(DE). We have Theorem 2.3.7 iv) wsint(D)⊆wsint(DE) and wsint(E)⊆wsint(DE). This implies that wsint(D)wsint(E) ⊆wsint(DE).

**2.5 ws-Neighbourhood of topological spaces**

**Definition 2.5.1:** Let (P, τ) is a topological space and xP, D subset N of P is termed to be ws- neighbourhood of x if ws-open set G ∋: xG⊆N



**Definition 2.5.2:** If (P, τ) is a topological space and D be a subset of P, a subset N of P is said to be ws- neighbourhood of D if their exists ws-open set G ∋: D⊆G⊆N.

**Definition 2.5.3:** The set of all ws-neighbourhood of xP is called ws-neighbourhood system at x and it is indicated as ws-N(P)



**Theorem 2.5.4:** Each neighbourhood N of x P is a ws-neighbourhood of P.

**Proof:** Take up N be a neighbourhood of xP. To verify that N is a ws-neighbourhood of x by neighbourhood definition an open set G ∋: xG⊆N. Hencefort N is a ws-neighbourhood of x.



**Remark 2.5.5:** As usually, a ws-neighbourhood N of x belongs to P need not be a neighbourhood of x in P.

**2.6 ws- Limit points**

**Definition 2.6.1:** Let (P, τ) be a topological space and D be a subset of P, then a point xP is called a ws-limit point of D iff each ws-neighbourhood of x contains a point of D distinct from x that is ((N-{x})∩D)≠φ for each ws-neighbourhood N of x.

Also equivalently Every ws-open set G containing x contains a point of D which is other than x.

**Definition 2.6.2:** The set of all ws-limit points of the set D is termed a derived set D and is indicated by wsd(D)

**Theorem 2.6.3:** Each neighbourhood N of x belongs to P is a ws-neighbourhood of P.

**Proof:** Take up N be a neighbourhood of xP. To verify that N is a ws-neighbourhood of x by the definition of neighbourhood an open set G ∋: xG⊆N. Henceforth N is a ws-neighbourhood of x.

**Remark 2.6.4:** As usually, a ws-neighbourhood N of x belongs to P need not be a neighbourhod of x in P as seen from the following example.

**Example 2.6.5:** Let P = {1, 2, 3, 4}, τ = {φ, P, {1}, {2}, {1, 2}, {1, 2, 3}}. Then wsO(P)= { φ, P, {1}, {2}, {3}, {4},{1, 2},{2, 3}, {3, 4}, {1, 2, 3}}. The set {2.3, 4}is a ws- neighbourhood of the point d. Seeing that the ws-open set {3, 4} is ∋: {4}⊆{3, 4} ⊆{2, 3, 4}. However the set {2, 3, 4} is not a neighbourhood of the point 4, since no open set G exists ∋: 3G⊆{3, 4}

**CHAPTER -3**

**On ws-Continuous and ws-Irresolute Maps in Topological Spaces**

**3.1 Introduction**

The concept of continuous functions plays a very important role in general topology. The regular continuous and completely continuous functions are introduced and studied by Arya S P [2]. Later, R S wali et all [31] introduced and investigated rw-continuous functions in topological spaces.

Second part of this chapter, a new class of maps called weakly semi closed-continuous (briefly, ws-continuous) maps are introduced and analyzed their properties with other existing generalized continuous maps.

Third part of this chapter, we analyse the concepts of ws-irresolute maps and strongly ws-continuous maps in topological spaces and investigate some of their properties.

**3.2 ws-continuous maps and some of their properties.**

**Definition 3.2.1:** A function h of topological space P in to a topological space Q is called a ws-continuous if inverse image of each closed set in Q is a ws-closed in P.

**Example 3.2.2:** Let P=Q= {1, 2, 3}. Let = {, P, {1}, {2}, {1, 2}} be a topology on P and = {, Q {1}, {2}, {1, 2}, {1, 3}} be a topology on Q and wsC(P) = {, P, {1}, {2},{3}, {1, 3},{2, 3}}. Let h: PQ defined by identity function is ws-continuous.

**Theorem 3.2.3:** Each continuous function is ws-continuous but inverse is untrue.

**Proof:** Take up h:(P,)(Q )be continuous . F is any closed in Q. Then (F) is closed set in P. Since each closed in P is ws-closed then (F) is ws-closed set in P. Therefore h is ws-continuous.

**Example 3.2.4:** Let P=Q= {1, 2, 3}. Let = {, P, {1}, {2}, {1, 2}} be a topology on P and = {, Q {1}, {2}, {1, 2}, {1, 3}} be a topology on Q and wsC(P) = {, P, {1}, {2},{3}, {1, 3},{2, 3}}. Let h: (P,)(Q ) be a function defined by identity function is ws-continuous but not a continuous function as closed set F={2} in Q, (F)={2} is not exist in closed set of P.

**Theorem 3.2.5:** If a map h: PQ is continuous then the following holds.

1. Each semi-continuous is ws-continuous but inverse is untrue.
2. Each -continuous is ws-continuous but inverse is untrue.
3. Each g#-continuous is ws-continuous but inverse is untrue.
4. Each \*g-continuous is ws-continuous but inverse is untrue.
5. Each g\*-continuous is ws-continuous but inverse is untrue.
6. Each gp-continuous is ws-continuous but inverse is untrue.
7. Each -continuous is ws-continuous but inverse is untrue.
8. Each regular-continuous is ws-continuous but inverse is untrue
9. Each rb-continuous is ws-continuous but inverse is untrue.

Proof: Proof can be obtained from the evidence that each semi (respectively, , g#, \*g-closed, g\*, gp, , regular and )closed is ws-C(P).

**Example 3.2.6:**  In example **3.2.4**, h is ws-continuous but not a semi(respectively, , g#, \*g, g\*, gp, , regular and ) continuous as the closed set F={2} in Q, (F)={ 2} is not a semi-closed (respectively, , g#, \*g, g\*, gp, , regular and ) closed in P.

**Theorem 3.2.7:** Each g#s- continuous is ws-continuous but inverse is untrue.

**Proof**: Proof obtained from the evidence that each g#s -C(P) is ws-C(P).

**Example 3.2.8:** Let P=Q= {1, 2, 3, 4}. Let = {, P, {1}, {2}, {1, 2},{1,2,3}}be a topology on P and ={, Q,{1,2},{3,4}}be a topology on Q and wsC(P)={P, {1},{2},{3},{4},{1,3},{1,4},{2,3},{2,4},{3,4},{1,2,4},{1,3,4},{2,3,4}},g#sC(P)={P,,{2},{3},{4},{2,3},{2,4},{3,4},{2,3,4}}. Let h: (P, )(Q, ) be a function defined by h(1)= 1, h(2)=4, h(3)=3, h(4)=2, is ws-continuous but not a g#s- continuous function as the closed set F={1,2} in Q, (F)={1,4} is not a g#s- closed in P.

**Theorem 3.2.9:** Map h: PQ is continuous then the following holds.

1. If h is ws-continuous then it is gspr-continuous but inverse is untrue.
2. If h is ws-continuous then it is gsp- continuous but inverse is untrue.
3. If h is ws-continuous then it is rgb-continuous but inverse is untrue.

**Proof:**

1. Take up F be a closed in Q. seeing that h is ws-continuous then (F) is ws-closed in P. Also each ws- closed is gspr-closed, then (F) is gspr-closed in P. Henceforth h is gspr- continuous.

Similarly we prove (ii) and (iii).

**Example 3.2.10:** Let P=Q= {1, 2, 3, 4}. Let ={, P, {1,2}, {3,4 }}be a topology on P and ={, Q, {1}, {2}, {1, 2},{1,2,3}} be a topology on Q and wsC(P)={P, {1,2}, {3,4}}, Let h: (P, )(Q, ) be a function defined by h(1)= 1, h(2)=2, h(3)=3, h(4)=4, is gspr-continuous, gsp- continuous and rgb-continuous but not a ws-continuous function as the closed set F={4} in Q, (F)={4} is not a ws-closed in P.

**Remark 3.2.11:**  Following example shows that ws-continuos is independent of g-continuous, w-continuous, -continuos, gp-continuos, g-continuos, swg-continuos, rwg-continuous, wg-continuous, g\*p-continuous, gw-continuos, \*\*g-continuous, swg\*-continuous, gr-continuous, w-continuous, g\*-continuous, rg-continuous, rg-continuos, R\*-continuous, rgw- continuous, wgr-continuos, gprw- continuous, pgr-continuos, g\*\*-continuous, wg\*\*-continuous, mildly g-continuous.

**Example 3.2.12:** Let P= Q = {1, 2, 3}. Let = {, P, {1}, {2}, {1, 2}}be a topology on P and ={ Q, {1},{2,3}} be a topology on Q and wsC(P)={P, ,{1}, {2}, {3},{2,3},{1, 3}}. Let h: (P, )(Q, ) be a function defined by identity function is ws-continuous but not g-continuous, w-continuous, -continuos, gp-continuos, g-continuos, swg- continuos, rwg-continuous, wg-continuous, g\*p-continuous, gw-continuos, \*\*g-continuous, swg\*-continuous, gr-continuous, w-continuous, g\*-continuous, rg-continuous, rg-continuos, R\*-continuous, rgw- continuous, wgr-continuos, gprw- continuous, pgr-continuos, g\*\*-continuous, wg\*\*-continuous, mildly g-continuous as the closed set H={1} in Q, (H)={1} is not a g(respectively w, , gp, g, swg, rwg, wg, g\*p, gw,\*\*g, swg\*, gr, w, g\*, rg, rg, R\*, rgw, wgr, gprw, pgr, g\*\*, wg\*\*, mildly g)closed set in P.

**Example 3.2.13:** Let P=Q= {1, 2, 3}. Let = {, P, {1}, {2,3}}be a topology on P and ={, Q, {1},{2}, {1,2}} be a topology on Q and wsC(P)={, P, {1},{2,3}}. Let h: (P, )(Q, ) be a function defined by identity function is g-continuous, w- continuous, -continuos, gp-continuos, g-continuos, swg-continuos, rwg-continuous, wg-continuous, g\*p-continuous, gw-continuos, \*\*g-continuous, swg\*-continuous, gr-continuous, w-continuous, g\*-continuous, rg- continuous, rg-continuos, R\*-continuous, rgw- continuous, wgr-continuos, gprw- continuous, pgr-continuos, g\*\*-continuous, wg\*\*-continuous, mildly g-continuous but not ws-continuous as the closed set F={3} in Q, (F)={3} is not a ws-C(P).

**Remark 3.2.14:** From the above discussions and known facts, the relation between ws-continuous and some existing continuous functions in topological space is shown in the following figure.

g-continuous, w-continuous, -continuos, gp-continuos, g-continuos, swg-continuos, rwg-continuous, wg-continuous, g\*p-continuous, gw-continuos, \*\*g-continuous, swg\*-continuous, gr-continuous, w-continuous, g\*-continuous, rg-continuous, rg-continuos, R\*-continuous, rgw- continuous, wgr-continuos, gprw- continuous, pgr-continuos, g\*\*-continuous, wg\*\*-continuous, mildly g-continuous

Regular Continuous

Continuous

Semi-Continuous

-Continuous

-Continuous

gp-Continuous

\*g-Continuous

gspr-Continuous

gsp-Continuos

rgb-Continuous

Ws-Continuous

g\*-Continuous

CONTINUOS

rb-Continuous

Note: A B Means A implies B, but inverse is untrue.

A B Means A and B are independent.

**Theorem 3.2.15:** Pretend h: PQ is a map. Then the statements given below are one and the same..

1. h is ws-continuous
2. The inverse image of every open set in Q is ws-open in P.

**Proof:**

1. Take up h: PQ be a ws-continuous. Let M be an open set in Q, then Mc is closed in Q. Since h is ws-continuous, h(Mc) is ws-closed in P. But (Mc)=P (M). Thus (M) is ws-open in P.
2. Suppose that inverse image of every open set in Q is ws-open in P. Let V be any closed set in Q. By hypothesis (Vc) is ws-open in P. But (Vc) = P (V). Thus P (V) is ws-open in P, so (V) is ws-closed in P. Thus h is ws-continuous.

**Theorem 3.2.16:** If h: (P,)(Q, ) is a map then the following holds.

1. h is ws-continuous and contra r-irresolute map then h is ws-continuous.
2. h is semi-continuous and contra continuous map then h is ws-continuous.
3. h is -continuous and contra irresolute map then h is ws-continuous.
4. h is ws-continuous and contra r-irresolute map then h is regular continuous.
5. h is \*g-continuous and contra continuous then h is ws-continuous.
6. h is gp-continuous and contra irresolute then h is ws-continuous.
7. h is gp-irresolute then it is ws-continuous.

**Proof:**

1. Take up D is any regular closed set of Q. Seeing that each regular-closed is closed, D is closed set in Q. Since h is ws-continuous and contra r-irresolute map, (V) is ws-closed and regular open in P. Now by results 1.3.3[5] (D) is semi-closed in P. Thus h is semi-continuous.
2. Take up D be closedset of Q. Since h is -continuous and contra continuous,(D) is semi-closed and open in P. Now by results 1.3.3 [5] (V) is ws-C(P) in P. Thus h is ws-continuous.

Similarly we prove (iii), (iv), (v), (vi) and (vii)

**Theorem 3.2.17:**  If h: PQ is ws-continuous then h(wscl(D))cl(h(D)) for each subset D of P but inversely is untrue.

**Proof:** Let h: PQ be ws-continuous. Let D be subset of P. Then cl(h(D)) is closed in Q, this implies ((cl(h(D)) is ws-closed in P. Also h(D)cl(h(D)) and D(cl(h(D)). Hence wscl(D)(cl(h(D))). Therefore h(wscl(D))cl(hD)).

**Example 3.2.18:**  Let P=Q={1,2,3}, ={, P,{1},{2},{1, 2}}, ={, Q, {2},{3}, {1, 3}, {2, 3}} and wsC(P)= {, P, {1},{2},{3}, {1, 3},{2, 3}}. Let h: (P,)(Q, ) be a function defined by identity function, For each subset D of P, h(wscl(A))cl(h(A)) holds. But h is not a ws-continuous as the closed set V= {1, 2} in Q, (V) = {1, 2} is not a ws-closed in P.

**Theorem 3.2.19:** Let h: (P,)(Q, ) be a map. Then the following statements are one and the same.

1. xP and every open set V in Q with h(x)V, a ws-open set U in P ∋: xU and h(U)V.
2. For each subset D of P, h(wscl(D))cl(h(D))
3. For each subset E of Q, wscl((E)) (cl(E))

**Proof:**

(i)(ii)

Pretend (i) holds and let y h(wscl(D)) and V be an open set containing Q. From (i), x wscl(D) h(x)=y and ws-open set M containing x h(M)V and x wscl(D).Then by theorem 3.2.17, M. That is h(MD)h(M)h(D)Vh(D). Therefore h(wscl(D))cl(h(D)).

(ii)(i)

Pretend (ii) holds and V is an open set in Q containing h(x). Let D(Vc). This implies that xD. Since h(wscl(D))cl(h(D))Vc. This implies that wscl(D)(Vc)=D. Since xD implies that xwscl(D) and by theorem 3.2.18, a ws-open set M containing x ∋: M then MDc and hence h(M)h(Dc)V.

(ii)(iii)

Pretend (ii) holds. Take up E is any subset of Q. Changing D by (E) in (ii) we get h(wscl((E)))cl(h((E)))cl(E). Hence wscl((E)) (cl(E)).

(iii)(ii)

Pretend (iii) holds. Let E=h(D) where D P. Then from (iii) we get wscl((h(D))) (cl(h(D))). That is wscl(D) (cl(h(D))). Therefore wscl((E)) (cl(E)).

**Definition 3.2.20:** Let (P, ) be a topological space and ws ={VP / wscl(Vc)=Vc}is a topology on P.

**Definition 3.2.21:** A topological space (P,) is called a Tws space if each ws-closed is closed.

**Definition 3.2.22:** A topological space (P,) is called a wsTsclspace if each ws-closed is semi-closed in P.

**Remark 3.2.23:** Composition of two ws-continuous maps is not continuous.

**Example 3.2.24:** Take up P=Q=R= {1, 2, 3}. Let = {, P, {1}, {2}, {1, 2}, {1, 3}} be a topology on P, = {, Q, {1},{2}, {1, 2}} be a topology on Q and ={, R, {1}, {2, 3}} be a topology on R. wsC(P)= {, P, {2}, {3}, {1, 3},{2,3}}, wsC(Q)= {, Q, {1},{2}, {3}, {1, 3},{2,3}}, Let h: (P, )(Q, ), g: (Q, )(R, ) and goh: (P, )(R, ) are identity functions. Both h and g are ws-continuous but goh is not a ws-continuous map as the closed set F={1} in R, (goh) -1 (F)={1} is not ws-C(P) in P.

**Theorem 3.2.25:** Let h: PQ is ws-continuous and g: QR is continuous then goh: PR is ws-continuous.

**Proof**: Take up V is any open set in R. Seeing that g is continuous, (V) is open in Q. Since h is ws-continuous, ((V)) =(V) is ws-open in P. Hence goh is ws-continuous.

**Theorem 3.2.26:** Let h: PQ and g: QR be ws-continuous functions and Q be Tws space then goh: PR is ws-continuous.

**Proof**: Take up V be any open set in R. Seeing that g is ws-continuous, (V) is ws-open in Q and Q is Tws space, then (V) is open in Q. Seeing that h is ws-continuous ((V)) =(V) is ws-open in P. Hence goh is ws-continuous.

**Definition 3.2.27:**Map h: PQ is called a perfectly ws-continuous if (V) is clopen (open and closed) set in P for each ws-open set V in Q.

**Theorem 3.2.28:** If h: PQ is continuous then .

1. If h is perfectly ws-continuous then it is ws-continuous
2. If h is perfectly ws-continuous then it is gspr(resp.gsp and rgb)continuous

**Proof:**

1. Take up M be open set in Q. Seeing that h is perfectly continuous then (M) is both open and closed in P. Also each open is ws-open, (M) is ws-open in P. Hence h is ws-continuous.

Similarly we can prove ii)

**Definition 3.2.29:** A function h: PQ is called ws\*-continuous if (V) is ws-C(P) in P for every semi-closed set V in Q.

**Theorem 3.2.30:** If h: PQ is ws\*-continuous then it is ws-continuous but inverse is untrue.

**Proof:** Take up h: PQ be ws\*-continuous. Pretending F be any closed set in Q. Seeing that h is ws\*-continuous and (F) ws-C(P). Since each closed set is semi- closed in Q then the inverse image (F) is ws-closed in P. Henceforth h is ws-continuous.

**Example 3.2.31:** Take up P=Q= {1, 2, 3}. Let = {, P, {1}, {2}, {1, 2},{1,2}}be a topology on P and ={, Q, {1}, {2},{1,2}} be a topology on Q and wsC(P)= {, P, {2},{3}, {1, 3},{2,3}}, SCL(Q)= {, Q, {1}, {2},{3}, {2, 3},{1,3}}. Let h: (P, )(Q, ) be a function defined by identity function will be ws-continuous but not a ws\*-continuous function as the semi closed set F={1} in Q, (F)={1} is not a ws-closed in P.

**3.3 ws- Irresolute and strongly ws-continuous functions**

**Definition 3.3.1:** Map h: PQ is called a ws-irresolute map if (V) is ws-closed for every ws-closed set V in Q.

**Definition 3.3.2:** Map h: PQ is called a strongly ws-continuous map if (V) is C(P) for every ws-closed set V in Q.

**Theorem 3.3.3:** If h: (P,)(Q, ) is ws-irresolute then it is ws-continuous but inverse is untrue.

**Proof:** Take up h: PQ be ws-irresolute. Prerend F be any closed set in Q and hence ws-closed in Q. Since h is ws-irresolute, (V) is ws-closed in P. Therefore h is ws-continuous.

**Example 3.3.4:** Take up P=Q= {1, 2, 3}. Let ={, P, {1}, {2}, {1, 2}, {1, 3}}be a topology on P and ={, Q, {1}, {2}, {1, 2}} be a topology on Q and wsC(P) ={, P, {2},{3}, {1, 3},{2, 3}}, wsC(Q)={, Q, {1}, {2},{3}, {1, 3},{2, 3}}. Let h: (P, )(Q, ) be a function defined by identity function is ws-continuous but not a ws-irresolute map as the ws-closed set F={1} in Q, (F)={1} is not a ws-C(P) in P.

**Theorem 3.3.5:** If h: (P, )(Q, ) is ws-irresolute iff (V) is ws-open set in P open set V in Q.

**Proof:** Pretend that h: PQ is ws-irresolute and U be ws-open set in Q. hence Mc is ws-closed in Q. By the definition of ws-irresolute, (Mc) is ws-closed in P. But (Mc)= P (M). Thus h (M) is ws-open in P.

Inversely,

Pretend that (H) is ws-open set in P for each ws-open set H in Q. Assume H be any ws-closed in Q. By the definition, (Hc) is ws-open in P. But (Hc)= P (H). Thus P (H) is ws-open in P and hence (H) is ws-closed in P. Therefore h is ws-irresolute.

**Theorem 3.3.6:**  If h: (P,)(Q, ) is ws-irresolute then it is ws\*-continuous but inverse is untrue.

**Proof:** Take up h: PQ be ws-irresolute. Assume H be any semi closed in Q. Since each semi closed is ws-closed and hence H is ws-closed in Q. By the definition of ws-irresolute, (H) is ws-C(P) . Therefore h is ws\*-continuous.

**Example 3.3.7:** Take up P=Q= {1, 2, 3}. Let ={, P, {1}, {2}, {1, 2}, {1, 3}}be a topology on P and ={, Q, {1}, {2}, {1, 2}} be a topology on Q and wsC(P)= ={, P, {2},{3}, {2, 3},{2, 3}}, wsC(Q)= {φ, Q, {1}, {2},{3}, {1, 3},{2, 3}}. Let h: (P, )(Q, ) be a function defined by identity function is ws\*-continuous but not a ws-irresolute map as the ws-closed set H={1} in Q, (H)={1} is not a ws-closed in P.

**Theorem 3.3.8:**  Let h: (P,)(Q, ) is ws-irresolute then h(wscl(D))scl(h(D)) for each subset D of P.

**Proof:** Take up DP and scl(f(D)) is ws-closed in Q. Since h is ws-irresolute (wscl(D)) is ws-closed in P. Further D(h(D))(scl(h(D))). By the definition of ws-closure, wscl(D)cl(D)). Hence h(wscl(D))scl(h(D)).

**Theorem 3.3.9:**  Let h: (P, )(Q, ) and g: (Q, )(R, ) be any two functions. Then

1. goh: (P, )(R, ) is ws-irresolute if semi is ws-irresolute and h is ws-irresolute.
2. goh: (P, )(R, ) is ws-continuous if semi is ws-continuous and h is ws-irresolute.

**Proof:** (i) Take up H be any ws- closed in (R,). Since semi is ws-irresolute then (H) is ws-closed in (Q,). Since h is ws-irresolute ((H)) is ws-closed in (P,). But (H)= ((H)) and hence goh is ws-irresolute.

(ii) Take up H as ws-closed in (R,). Since semi continuos is ws-continuous then (H) is ws-closed in (Q, ). Seeing that h is ws-irresolute ((H)) is ws-closed in (P,). But (H)=((H)) and hence goh is ws-continuous.

**Theorem 3.3.10:** If h: (P,)(Q, ) is strongly ws-continuous then h is continuous but inverse is untrue.

**Proof:** Take up h: PQ be a strongly ws-continuous and H be any closed set in Q. Seeing that each closed set is ws-closed set and hence H is ws-closed in Q. Also h is strongly ws-continuous then (H) is closed set in P. Therefore h is continuous.

**Example 3.3.11:** Take up P=Q= {1, 2, 3}. Let = {, P, {1}, {2}, {1, 2},{1,3}}and ={, Q, {1}, {2}, {1, 2}} and wsC(P)= {, P, {2}, {3}, {2, 3},{2,3}}, wsC(Q) = {, Q, {1}, {2},{3}, {1, 2},{2,3}}. Let h: (P, )(Q, ) be a function defined by identity function is continuous but not strongly ws-continuous as the ws-closed set H={1} in Q, (H)={1} is not a closed set in P.

**Theorem 3.3.12:** Each strongly ws-continuous is strongly semi-continuous but inverse is untrue.

**Proof:** Take up h: PQ be strongly ws-continuous. Assume F be any semi –closed set in Q. Seeing that each semi-closed is ws-closed and hence H is ws-closed in Q. Seeing that h is strongly ws-continuous then (H) is closed set in P and hence semi-closed in P. Therefore h is semi -continuous.

**Example 3.3.13:** Take up P= Q = {1, 2, 3, 4}. Let = {, P, {1}, {2}, {1, 2},{1,2,3}} and ={, Q, {1}, {1,2}, {1,2,3}} and SC(Q)= {, Q, {2}, {3}, {4},{2, 3},{2, 4}, {3, 4}, {2, 3,4}}, wsC(Q)= {, Q, {2}, {3}, {4},{1,4},{2,3},{2,4},{3,4},{1,2,4},{1,3,4},{2,3,4}}, Let h: (P, )(Q, ) be a function defined by h(1)=1, h(2)=4, h(3)=4, h(4)=4, is continuous but not strongly ws-continuous as the ws-closed set H={1,4} in Q, (H)={1,4} is not a closed in P.

**Theorem 3.3.14:** If a mapping h: (P,)(Q, ) is strongly ws-continuous iff (M) is open set in P for each ws-open set M in Q.

**Proof:** Suppose that h: PQ is strongly ws-continuous. Let M be any ws-open set in Q and hence Mc is ws-closed set in Q. Seeing that h is strongly ws-continuous (M) is closed in P. But (Mc) =P (M). Thus (M) is open in P.

Inversely suppose that (M) is open set in P for each ws-open set M in Q. Assume H is any ws-closed in Q and hence Hc is ws-open in P. But (Hc) =P (H). Hence P (H) is open in P and so (H) is closed in P. Therefore h is strongly ws-continuous.

**Theorem 3.3.15:** Each strongly continuous is strongly ws-continuous but inverse is untrue.

**Proof:** Take up h: PQ is strongly continuous, Assume G be any ws-open set in Q and also any subset of Q. Seeing that h is strongly continuous then (G) is both open and closed in P, say (G) is open in P. Therefore h is strongly ws-continuous.

**Example 3.3.16:** Take up P=Q= {1, 2, 3, 4}. Let = {, P, {1}, {2}, {1, 2},{1,2,3}} and ={, Q, {1}, {1,2}, {1,2,3}} and C(Q)= {, Q,{4},{3,4},{2,3,4}}, wsC(Q)= {,Q,{2},{3}, {4},{1,4},{2,3},{2,4}, {3,4}, {1,2,4},{1,3,4},{2,3,4}}, Let h: (P, )(Q, ) be a function defined by h(1)=4, h(2)=4, h(3)=4, h(4)=4 is strongly ws-continuous but not a strongly continuous as the set H={4} in Q,(H)={4} is not a clopen set in P.

**Theorem 3.3.17:** Each strongly ws-continuous is ws-continuous but inverse is untrue.

**Proof:** Take up h: PQ be strongly ws-continuous. Let H be any closed in Q and hence ws-closed in Q. Seeing that h is strongly ws-continuous then (H) will be closed in P and hence ws-closed in P. Therefore h is ws-continuous.

**Example 3.3.18:** Take up P=Q= {1, 2, 3}. Let = {, P, {1}, {2}, {1, 2}, {1, 3}} and = {, Q, {1}, {2}, {1,2}} and wsC(P)= {, P, {2}, {3}, {2, 3},{2,3}}. Let h: (P,)(Q, ) be a function defined by h(1)=1, h(2)=1, h(3)=3 ws-continuous but not strongly ws-continuous as the ws-closed set H= {3} in Q, (H)={3} is not a closed set in P.

**Theorem 3.3.19:** In discrete topological space, each strongly ws-continuous is strongly continuous.

**Proof:** Take up h: PQ be strongly ws-continuous in a discrete topological space. In view that H be any subset of Q. Seeing that H is both open and closed subset of Q in discrete space. We get the following two cases.

Case (i) Take up H is any closed subset of Q and hence ws-closed in Q. seeing that h is strongly ws-continuous then (H) is closed in P.

Case (ii) Take up H be any open subset of Q and hence ws-open in Q. Seeing that h is strongly ws-continuous then (H) is open in P.

Therefore (H) is both open and closed in P. Hence h is strongly continuous.

**Theorem 3.3.20:** Pretend h: PQ and g: QR be any two functions. Then

1. goh: PR is strongly ws-continuous if both h and g are ws-continuous.
2. goh: PR is strongly ws-continuous if g is strongly ws-continuous and h is continuous.
3. goh: PR is ws-irresolute if g is strongly ws-continuous and h is ws-continuous.
4. goh: PR is continuous if g is ws-continuous and h is strongly ws-continuous.

**Proof:**

1. Take up G be ws-closed in (R,). Seeing that g is strongly ws-continuous then (G) is closed set in (Q,) and hence ws-closed set in (Q,). And h is also strongly ws continuous then ((G)) closed set in (P,). But (G) = ((G)) and hence goh is strongly ws-continuous.
2. Take up G be ws-closed set in (R,). Seeing that g is strongly ws-continuous then (G) is closed set in (Q,). Also as h is continuous then ((G)) is closed set in (P,). But (G) = ((G)) and hence goh is strongly ws-continuous.
3. Take up G be any ws-closed set in (R, ). Seeing that g is strongly ws-continuous then (G) is closed set in (Q,).Also as h is ws-continuous then ((G)) is ws-closed in (P,). But (G) = ((G)). Hence goh is ws-irresolute.
4. Take up G be any closed set in (R,). Seeing that g is ws-continuous then (G) is ws-closed in (Q,). Since h is strongly ws continuous then ( (G)) closed in (P,). But (G) = ((G)). Hence goh is continuous.

**Theorem 3.3.21:** Take up h: PQ and g: Q R be two functions. Then

1. goh: P R is strongly ws-continuous whenever g is perfectly ws-continuous and h is continuous.
2. goh: PR is perfectly ws-continuous whenever g is strongly ws-continuous and h is perfectly ws-continuous.

**Proof:**

1. Take up G be any ws-open set in (R,). Seeing that g is perfectly ws-continuous then (G) is clopen set in (Q,), say (G) is open set in (Q,). Since h is continuous then ((G)) open set in (P,). Thus (G) = ((G)). Hence goh is strongly ws-continuous.
2. Take up G be a ws-open set(R, ). Seeing that g is strongly ws-continuous then (G) is open set (Q,). Since h is perfectly ws-continuous then ((G)) clopen set in (P,). But (G) = ((G)). Hence goh is perfectly ws-continuous.

**Theorem 3.3.22:** Pretend (P, ) be a discrete topological space and (Q,) be any topological space. Let h: (P,)(Q, ) be a function. Then the statements given below are one and the same..

1. h is strongly ws-continuous
2. h is perfectly ws-continuous.

**Proof:**

(i)(ii)

Take up G be any open set in (Q,). Seeing that h is strongly ws-continuous then (G) is open set in (P,). But in discrete space, (G) is closed (P,). Thus (G) is both open and closed (P,). Therefore h is perfectly ws-continuous.

(ii)(i)

Take up M be any ws-open set in (Q,). Seeing that h is perfectly continuous then (G) is both open and closed set in (P,). Hence h is strongly ws-continuous.

**Theorem 3.3.23:** Pretend (P,) be any topological space and (Q, ) be Tws space and h: (P,)(Q, ) be a map. Then the results given below are one and the same..

1. h is strongly ws-continuous
2. h is continuous

**Proof:**

(i)(ii)

Take up H be any closed set (Q,). Seeing that each closed set is ws-closed and hence H is ws-closed in (Q,). Since h is strongly ws-continuous then (H) is C(P) in (P,). Hence h is continuous.

(i)(ii)

Take up G be any ws-closed in (Q, ). Since (Q, ) is Tws space, H is closed in (Q,). Seeing that h is continuous implies (H) is closed in (P,). Hence h is strongly ws-continuous.

**Theorem 3.3.24:** Let h: (P,)(Q, ) be a map. Both (P,) and (Q,) are Tws space. Then the results given below are one and the same..

1. h is ws-irresolute.
2. h is strongly ws-continuous.
3. h is continuous.
4. h is ws-continuous.

The proof is obvious.

**Theorem 3.3.25:** Take up P and Q be wsTsemi spaces. Then for h: (P,)(Q,) the results given below are one and the same..

1. h is sc-irresolute
2. h is ws-irresolute

**Proof:**

(i)(ii)

Take up h: PQ be sc -irresolute. Let H be a semi-closed in Q and thus ws-closed in Q. Seeing that h is sc -irresolute then (H) is semi-closed in P and thus ws-closed in P. Therefore h is ws-irresolute.

(i)(ii)

Take up h: PQ be ws-irresolute. Let H be a semi-closed in Q and thus ws-closed in Q. Seeing that h is ws-irresolute then (H) is ws-C(P) in P. But P is wsTsemi space and hence (H) is semi-closed in P. Therefore h is semi-irresolute.

**CHAPTER-4**

**On ws closed and ws open Maps in Topological Spaces**

**4.1. Introduction**

In 1982, the concept of generalized closed maps is introduced and studied by S. R, Malghan [23], wg-closed maps and rwg-closed maps were introduced and studied by Nagavani [26]. Later regular closed maps, rw-closed maps and αrw-closed maps have been introduced and studied by Long [20], Benchalli et.all [8] and R S Wali et.all [35] respectively.

In section 2 of this chapter, a new class of maps called ws-closed maps is introduced and their relations with various generalized closed maps are studied. We prove that the composition of two ws-closed maps need not be ws-closed map. We also introduce ws\*-closed maps, ws-open maps and ws\*-open maps in topological spaces and obtain certain characterizations of these maps.

**4.2 ws–closed Maps and ws–open Maps**

**Definition 4.2.2:** A map h: P → Qis said to be weakly semi-closed (briefly ws-closed) map if the image of every closed set in (P, τ) is ws–closed in (Q, σ).

**Example 4.2.1:** Let P = Q = {1, 2, 3}, τ = {P, ϕ, {1}, {2}, {1, 2}} be a topology on P. σ = {Q, ϕ, {1}, {2, 3}} be a topology on Q and wsC(Q) ={Q, ϕ, {2},{3},{1,3},{2, 3}}. Let h: P → Qdefined by identity map, then h is ws–closed map.

**Theorem 4.2.3**: Each closed map is ws–closed map, but inverse is untrue.

**Proof:** Take up h: (P,)(Q, ) be closed map and V be any C(P) in P. Then h (V) is closed in Q, Seeing that each closed set is ws–C(P). Hence h(V) is ws-closed in Q. Therefore h is ws -closed.

We use example 4.2.4 to prove the inverse of theorem is untrue.

**Example 4.2.4: Take up** P = Q = {1, 2, 3} τ = {P, ϕ, {2},{3},{1,3},{2, 3}} be a topology on P. σ ={Q, ϕ, {1},{2},{1, 2}} be a topology on Q and wsC(Q) ={Q, ϕ,{1}, {2},{3},{1,3},{2, 3}}. Let h: P → Qdefined by identity map, hence h is ws–closed map but not closed as image of closed set {2} in P is {2}, is not closed in Q.

**Theorem 4.2.5:** If h: PQ is a closed map then the following holds.

1. Each semi-closed is ws-closed but inverse is untrue.
2. Each α -closed is ws-closed but inverse is untrue.
3. Each g#-closed is ws-closed but inverse is untrue.
4. Each \*g-closed is ws-closed but inverse is untrue.
5. Each g\*-closed is ws-closed but inverse is untrue.
6. Each gp-closed is ws-closed but inverse is untrue.
7. Each -closed is ws-closed but inverse is untrue.
8. Each regular-closed is ws-closed but inverse is untrue
9. Each rb-closed is ws-closed but inverse is untrue.

**Proof:** Proof is obtained from the evidence that each semi (respectively, α, g#-, \*g, g\*, , regular, rb) closed set is ws–closed.

We use example 4.2.6 to prove the inverse of theorem is untrue.

**Example 4.2.6:** Take up P = Q = {1, 2, 3} τ = {P, ϕ, {2}, {3},{1,3},{2, 3}} be a topology on P. σ ={Q, ϕ, {1}, {2}, {1, 2}} be a topology on Q and wsC(Q)={Q, ϕ,{1},{2},{3},{1,3},{2, 3}}. Let h: P → Q defined by identity map, hence h is ws–closed map but not closed as image of closed set {2} in P is {2}, is not semi- (respectively α , g#-, \*g-, g\*, , regular, rb) in Q.

**Theorem 4.2.7:**  Each g#s -closed map is ws–closed map, but inverse is untrue.

**Proof:** Proof of above theorem follows from the evidence that each g#s -C(P) is ws–C(P).

We use example 4.2.8 to prove the inverse of theorem is untrue.

**Example 4.2.8:** Take up P=Q= {1, 2, 3, 4}. Let = {, P, {1,2}, {3,4}}be a topology on P and ={φ, Q, {1}, {2}, {1, 2},{1,2,3}} be a topology on Q and wsC(Q)={Q, φ {1},{2},{3},{4},{1,3}, {1,4},{2,3},{2,4},{3,4},{1,2,4}, {1,3,4}, {2,3,4}}, g#sC(Q)={Q, φ {2},{3},{4},{2,3},{2,4},{3,4},{2,3,4}}. Let h: (P, )(Q, ) be a function defined by h(1)= 1, h(2)=4, h(3)=3, h(4)=4, is ws–closed map but not closed as image of closed set {1,2} in P is {1,4} is not g #s –closed set in Q.

**Theorem 4.2.9:** If h: PQ is a closed map then the following holds.

1. Each ws–closed map is gspr-closed map but inverse is untrue.
2. Each ws–closed map is gsp-closed map but inverse is untrue.
3. Each ws–closed map is rgb-closed map but inverse is untrue.

**Proof:** Proof is obtained from the evidence that each ws–C(P) is gspr C(P) (respectively, rgb-C(P), gsp-C(P)).

We use example 4.2.10 to prove the inverse of theorem is untrue.

**Example 4.2.10:**  Take up P=Q= {1, 2, 3, 4}. Let = {, P, {1}, {2}, {1, 2},{1,2,3}} be a topology on P and ={φ, Q, {1,2}, {3,4 } } be a topology on Q and wsC(Q)= {Q, ϕ, {1,2}, {3,4 }}, gsprC(Q)=gspC(Q)=rgbC(Q)={P(Q)}. Let h: (P, τ) → (Q,σ) defined by identity map, then h is gspr (respectively, rgb, gsp)closed map but not ws-closed as image of closed set {4} in P is {4} is not ws –closed in Q.

**Remark 4.2.11:**  The example given below shows that ws -closed maps are independent of pre, w, gp, g, , , g, gw , pgpr, gp, swg-closed, w ,g , -closed, g\*pre , mildly g , g\*p , wg , rg , gpr , wgr, pgr, \*\*g, rwg , wg\*\* , g\*\*, rg, gr, gpw , gw, g\* , R\* closed maps,

**Example 4.2.12:** Take up P=Q= {1, 2, 3, 4}. Let = {, P, {1},{1, 2},{1,2,3}}be a topology on P and ={, Q, {1,2}, {3,4 }} be a topology on Q and wsC(Q)= {Q, ϕ, {1,2}, {3,4 }}. Let h: P → Qdefined by, h(1) = 1, h(2) = 2, h(3) = 3, h(4)=4, then h is of pre, w, gp, g, , , g, gw , pgpr, gp, swg-closed, w ,g , -closed, g\*pre , mildly g , g\*p , wg , rg , gpr , wgr, pgr, \*\*g, rwg , wg\*\* , g\*\*, rg, gr, gpw , gw, g\* , R\* closed maps, But h is not ws–closed map, as closed set {d} in P is {d} is not ws–closed in Q.

**Example 4.2.13:**  Take up P=Q= {1, 2, 3, 4}. Let = {, P, {1,2}, {3,,4 }}be a topology on P and ={, Q, {1}, {2}, {1,2},{1,2,3} } be a topology on Q and wsC(Q)={, Q, {1},{2},{3},{4},{1,3}, {1,4},{2,3},{2,4},{3,4},{1,2,4}, {1,3,4}, {2,3,4}}. Take up h: P → Qdefined by, h(1) = 1, h(2) = 1, h(3) = 3, h(4)=4, then h is ws–closed, but not a pre(respectively, w, gp, g ,, , g, gw , pgpr, gp, swg, w ,g , , g\*pre , mildly g , g\*p , wg , rg , gpr , wgr, pgr, \*\*g, rwg , wg\*\* , g\*\*, rg, gr, gpw , gw, g\* , R\*) closed maps as C(P) {a , b} in P is {1} is not pre(respectively, w, gp,g ,,, g, gw , pgpr, gp, swg, w ,g , , g\*pre , mildly g , g\*p , wg , rg , gpr , wgr, pgr, \*\*g, rwg ,wg\*\* , g\*\*, rg, gr, gpw , gw, g\* , R\* ) closed set in Q

**Remark 4.2.14:** From the above discussions and known facts, the relation between ws-closed map and some existing closed maps in topological space is shown in the following figure

Regular closed map

Closed map

Semi-closed map

- Closed map

-Closed map

gp- closed map

\*g- Closed map

gspr- Closed map

gsp- Closed map

rgb- Closed map

g\*- closed map

CONTINUOS

rb- closed maps

pre–closed, w–closed, gp-closed,g-closed ,–closed,–closed, g–closed, gw –closed, pgpr–closed, gp–closed, swg-closed, w –closed ,g –closed, -closed, g\*pre –closed, mildly g –closed, g\*p –closed, wg –closed, rg –closed, gpr –closed, wgr–closed, pgr–closed, \*\*g–closed, rwg –closed,wg\*\* –closed, g\*\*–closed, rg–closed, gr–closed, gpw –closed, gw–closed, g\* –closed, R\*–closed maps

Ws-closed map

Note: A B Means A implies B, but inverse is untrue.

A B Means A and B are independent.

**Theorem 4.2.15:** Map h: P → Q is contra regular closed and gspr-closed map then h is ws-closed map in P.

**Proof:** Take up M as a closed set (P, τ). Then h (M) will be open and gspr –closed in Q. By results 1.3.3 h (M) is ws–closed in (Q, σ). Therefore h is ws–closed map.

**Theorem 4.2.16:** A map h: P → Qis contra closed and rgb-closed map then h is ws-closed map in P.

**Proof:** Take up M as a closed set (P, τ). Then h (M) will be open and rgb – closed in Q. By results 1.3.3 h (M) is ws–closed in (Q, σ). Therefore h is ws–closed map.

**Theorem 4.2.17:** Map h: P → Qis contra closed and αg–closed map then h is ws–closed map.

**Proof:** Take up M as a closed set (P, τ). Then h (M) will be open and αg– closed in Q. By results 1.3.3 h (M) is ws–closed in (Q, σ). Therefore h is ws–closed map.

**Theorem 4.2.18:** Map h: P → Qis contra closed and rgb-closed map then h is ws-closed map in P.

**Proof:** Take up M as a closed set (P, τ). Then h (M) will be open and rgb – closed in Q. By results 1.3.3 h(M) is ws–closed in (Q, σ). Therefore h is ws–closed map.

**Theorem 4.2.19:** Map h: P → Qis contra semi closed and swg\*-closed map then h is ws-closed map in P.

**Proof:** Take up M as a closed set (P, τ). Then h (M) will be semi open and swg\*– closed in Q. By results 1.3. h(M) is ws–closed in (Q, σ). Therefore h is ws–closed map.

**Theorem 4.2.20:** Map h: P → Qis contra semi closed and swg-closed map then h is ws-closed map in P.

**Proof:** Take up M as a closed set (P, τ). Then h (M) will be semi open and swg – closed in Q. By results 1.3.3 h (M) is ws–closed in (Q, σ). Therefore h is ws–closed map.

**Theorem 4.2.21:** Map h: P → Qis contra is semi closed and sg-closed map then h is ws-closed map in P.

**Proof:** Take up M as a closed set (P, τ). Then h (M) will be semi open and sg – closed in Q. By results 1.3.3 h (M) is ws–closed in (Q, σ). Therefore h is ws–closed map.

**Theorem 4.2.22:** Map h: P → Qis contra is semi closed and sgb-closed map then h is ws-closed map in P.

**Proof:** Take up M as a closed set (P, τ). Then h (M) will be semi open and sgb – closed in Q. By results 1.3.3 h (M) is ws–closed in (Q, σ). Therefore h is ws–closed map.

**Theorem 4.2**.**23**: Map h: P → Qis contra is semi closed and αgs-closed map then h is ws-closed map in P.

**Proof:** Take up M as a closed set (P, τ). Then h (M) will be semi open and αgs-closed in Q. By results 1.3.3 h (M) is ws–closed in (Q, σ). Therefore h is ws–closed map.

**Theorem 4.2.24:** Map h: P → Qis contra is β-closed and βwg\*-closed map then h is ws-closed map in P.

**Proof:** Take up M as a closed set (P, τ). Then h (M) is β- open and βwg\*-closed in Q. By results 1.3.3 h (M) is ws–closed in (Q, σ). Therefore h is ws–closed map.

**Theorem 4.2.25:** Map h: P → Qis contra is both closed and g-closed map then h is ws-closed map in P**.**

**Proof:** Take up M as a closed set (P, τ). Then h (M) will be open and g–closed in Q. By results 1.3.3 h (M) is ws–closed in (Q, σ). Therefore h is ws–closed map.

**Theorem 4.2.26:** Map h: P → Qis contra is regular semi closed and rw-closed map then h is ws-closed map in P**.**

**Proof:** Take up M as a closed set (P, τ). Then h (M) will be semi open and rw –closed in Q. By results 1.3.3 h (M) is ws–closed in (Q, σ). Therefore h is ws–closed map.

**Theorem 4.2.27**: Map h: P → Qis contra is regular semi closed and R\*-closed map then h is ws-closed map in P.

**Proof:** Take up M as a closed set (P, τ). Then h (M) will be semi open and R\*-closed in Q. By results 1.3.3 h (M) is ws–closed in (Q, σ). Therefore h is ws–closed map.

**Theorem 4.2.28:**  A mapping h: P → Qbe ws–closed, then wscl (h(D)) ⊆ h(cl (D)) subset D of (P, τ).

**Proof**: Pretend that h is ws-closed and D ⊆P. Then cl(D) is closed in P and so h(cl(D)) is ws–closed in (Q, σ). We have h(D) ⊆ h(cl(D)), wscl(h(D)) ⊆ wscl(h(cl(D))) → (i). Since h(cl (D)) is ws-closed in (Q, σ), wscl(h(cl(D))) = h (cl (D)) → (ii), From (i) and (ii), we have wscl (h(D)) ⊆ h(cl(D)) for each subset D of (P, τ).

**Corollary 4.2.29:**  A mapping h: P → Qbe ws- closed, then the image h(D) of closed set D in (P, τ) is τws– closed in (Q, σ).

**Proof:** Take up closed set D (P, τ). Since h is ws-closed, wscl (h (D)) ⊆ h (cl (D)) → (i). Also cl (D) = D, as D is a closed set and so h (cl(D)) = h(D) →(ii). From (i) and (ii), we have wscl (h (D)) ⊆ h (D). We know that h (D) ⊆ ws cl (h (D)) and so wscl (h (D)) =h (D). Therefore h (D) is τws-closed in (Q, σ).

**Theorem 4.2.30:** Take up (P, τ) be any topological spaces and Qbe a topological space where wscl (D) = α - cl (D) D of Q, and h: P → Qbe a map, and then given statements are one and the same..

1. h is ws–closed map.
2. wscl (h (D)) ⊆ h (cl (D)) subset D of (P, τ).

**Proof:**

(i) ⇒ (ii) obiuosly from Theorem 4.2.29.

(ii) ⇒ (i) Take up closed set D (P, τ). Then D=cl (D) and so h(D) = h(cl (D)) ⊇ wscl (h (D)) by hypothesis. We have h (D) ⊆ wscl (h (D)), Therefore h (D) = wscl (h (D)). Also h (D) = wscl (h (D)) = α-cl (h (D)), by hypothesis. That is h (D) = α-cl (h (D)) and

so h (D) is α-closed in (Q, σ). Thus h (D) is ws–closed Qand therefore h is ws–closed map.

**Theorem 4.2.31:** Map h : P → Q be ws-closed iff subset S of Q and each open set M containing (S) ⊆M, there is a ws–open set V of Q S ⊆V and (V ) ⊆ M.

**Proof:**  Pretend h is ws–closed. Assume S ⊆ Q and M is an open set P, (S) ⊆M. Now P − M will be closed in P. Seeing that h is ws-closed, h(P − M) is ws closed Q. Then V = Q – h (P − M) will be ws–open set Q. and it is known that (S) ⊆ M implies S ⊆ V and (V) =P − (h(P − M)) ⊆ P − (P − M) = M. That is (V)⊆ M.

Inversely,

Pretend H is C(P). Then ((h (H)) c ) ⊆ Hc and Hc iwill be open set P. From hypothesis, an ws–open set V Q h(H) c ⊆ V and (V) ⊆ Hc and so H ⊆ ((V)) c. Hence Vc ⊆ h(H) ⊆ h((((V))c ) ⊆Vc which implies h(H) = Vc . Seeing that V c is ws–closed, h (H) is ws–closed. Thus h (H) is ws–closed Qand therefore h is ws–closed map.

**Remark 4.2.32:** As usual, The composition of two ws–closed maps need not be ws–closed map and it is proved from the example given below.

**Example 4.2.33:** Take up P = Q = R = {1,2,3}, τ={P, ϕ, {1}, {2, 3}} be a topology on P, σ = {Q, ϕ, {1},{2},{1, 2}} be a topology on Q and η = {R, ϕ, {1}, {2}, {1, 2}, {1, 3}} be a topology on R. Define h: P → Q defined by h(1)=1, h(2)=2, h(3)=3 and g: Q→ (R, η) are the identity maps. Then h and g are ws–closed maps, but their composition g ◦ h : (P, τ) → (R, η) will not be ws–closed map, because H = {1} is closed in (P, τ), but g ◦ h (H) =g ◦ h ({1}) = g(h({1})) = g({1}) = {1} is not ws–closed in (R, η).

**Theorem 4.2.34:** If h: P → Qis closed map and g: Q→ (R, η) is ws–closed map, then the composition g ◦ h: (P, τ) → (R, η) is ws–closed map.

**Proof:**  Take up closed set F (P, τ). Seeing that h is closed map, h (F) will be closed in (Q, σ). Also g is ws–closed map, g (h(F)) will be ws–closed in (R, η). That is g ◦ h (F) = g (h (F)) is ws–closed and henceforth g ◦ h is ws–closed map.

**Theorem 4.2.35:** If h: P → Qand g: Q→ (R, η) is ws–closed maps and Qbe a Tws-space then g ◦ h: (P, τ) → (R, η) is ws–closed map.

**Proof**: Take up closed set D (P, τ). Seeing that h is ws–closed, h (D) is ws–closed in Q. From by hypothesis, h (D) is closed. Also g is ws–closed, g (h (D)) will be ws–closed in R and g(h(D)) = g ◦ h(D). Henceforth g ◦ h is ws–closed map.

**Theorem 4.2.36:** If h: P → Q is g-closed, g: Q → R, η be ws–closed and Q is T1/2-space then their composition g ◦ h: P → R is ws–closed map.

**Proof**: Take up closed set D (P, τ). Seeing that h is g-closed, h (D) will be g-closed in (Q, σ). Seeing that Qis T1/2-space, h(D) is closed (Q, σ). Since g is ws–closed, g (h(D)) is ws–closed R and g(h(D)) = g ◦ h(D). Henceforth g ◦ h is ws–closed map.

**Definition 4.2.37**: A map h: P → Q is called a ws–open map if the image h (D) is ws–open in Q open set D in (P, τ). From the definitions we obtain these following results.

**Theorem 4.2.38:**

* 1. Each open map is ws–open but inverse is untrue.
  2. Each semi-open map is ws–open but inverse is untrue.
  3. Each α -open map is ws–open but inverse is untrue.
  4. Each g#-open map is ws–open but inverse is untrue.
  5. Each \*g α -open map is ws–open but inverse is untrue.
  6. Each g\*-open map is ws–open but inverse is untrue.
  7. Each αgp-open map is ws–open but inverse is untrue.
  8. Each -open map is ws–open but inverse is untrue.
  9. Each rb-regular open map is ws–open but inverse is untrue
  10. Each g#s open map is ws–open but inverse is untrue.
  11. Each ws–open map is gspr-open but inverse is untrue..
  12. Each ws–open map is gsp-open but inverse is untrue..
  13. Each ws–open map is rgb-open but inverse is untrue..

**Theorem 4.2.39:** For each bijection map h: P → Q following results are one and the same.:

1. : Q → P is ws–continuous.
2. h is ws–open map
3. h is ws–closed map.

**Proof**:

(1) ⇒ (2) Take up a open set M P. By hypothesis (M) = h (M) is ws–open Qand so h is ws–open.

(2) ⇒ (3) Take up closed set H P. Then Hc is open set P. By hypothesis, h (H c) is ws–open Q. Also h (Hc) = h (H)c is ws–open Q and therefore h(H) is ws– closed in Q. Hence h is ws–closed.

(3) ⇒ (4) Take up closed set H P. By hypothesis, h(H) is ws–closed Q. But h (H) = (H) and henceforth is continuous.

**Theorem 4.2.40:** A Map h: P→ Q is ws–open, then h (int (D)) ⊆ ws–int (h(D)) for subset D P.

**Proof:**  Take up h: P→ Q be an open map and D is any subset P. Hence int(D) is open P and so h(int(D)) is ws–open Q. We have h(int(D)) ⊆ h(D). Henceforth h(int (D)) ⊆ ws–int (h(D)).

**Theorem 4.2.41:**  A map h: P→ Q is ws–open, then for each neighbourhood M of x P a ws– neighbourhood W and h(x) Q, :W ⊆ h(M).

**Proof.** Take up h: P→ Q is an ws–open map. Assume x∈P and M be an arbitrary neighbourhood of x P. Then an open set G P, : x ∈ G ⊆ M. Also h(x) ∈h (G) ⊆ h (M) and h (G) is ws–open set Q, as h is an ws–open map. And also by the result h (G) is ws– neighbourhood of each of its points. Taking h (G) = W, W is an ws–neighbourhood of h(x) in Q ∋: W ⊆ h (M).

**Definition 4.2.42:** A map h: P→ Q is termed as ws\*- closed map if the image h(D) is ws–closed Q for each ws–closed set D (P, τ).

**Theorem 4.2.43:** Each ws\*-closed map is ws–closed map but inverse is untrue..

**Proof.** Proof is obtained from the previous results, definations and evidence that each closed set is ws–closed.

**Example 4.2.44:** Take up P = Q = {1, 2, 3}, τ = {P, ϕ, {1}, {2}, {1, 2}} be a topology on P and σ = {Q, ϕ, {1}, {2},{1, 2},{1,3}} be a topology on Q and h: P → Q be the identity map. Thus h is ws–closed map but not ws\*-closed map. Since {1} is ws–closed in P but its image under h is {1},is not ws–closed in Q.

**Theorem 4.2.45:** If h: P → Qand g: Q→ (R, η) are ws\*-closed maps, then their composition g ◦ h: (P, τ) → (R, η) is also ws∗-closed.

**Proof**. Take up a ws- closed set H P. Seeing that h is ws\*– closed map, h (H) is ws–closed Q. Seeing that ws∗– closed map, g (h (H)) is ws–closed (R, η). Henceforth g ◦ h is ws\*–closed map. Analogous to ws\*-closed map, we define another new class of maps called ws\*–open maps which are stronger than ws–open maps.

**Definition 4.2.46:** A map h: P → Q is termed as ws∗– open map if the image h (D) is ws–open set in Q for each ws–open set D in P.

**Remark 4.2.47:** Seeing that each open set is a ws–open set, we have each ws∗–open map is ws–open map. As usual the inverse is untrue as seen from example given below.

**Example 4.2.48:** Take up P = Q = {1, 2, 3}, τ = {P, ϕ, {1}, {2}, {1, 2}} and σ ={Q, ϕ, {1}, {2}, {1, 2},{1,3}} and h: P → Q be the identity map. Then h is ws –open map but not ws\*-open map. Seeing that {2, 3} is ws–open set P, but its image under h is {2, 3} is not ws–open in Q.

**Theorem 4.2.49:** If h: P → Q and g: Q → R, are ws∗-open maps, then their composition g ◦ h: P → R, is also ws∗-open.

**Proof**. Proof is same as Theorem 4.2.45.

**Theorem 4.2.50:** For each bijection map h: P → Q then given results are one and the same:

1. : Q→ (P, τ) is ws irresolute.
2. h is ws∗–open map
3. h is ws∗–closed map.

**Proof:** Proof is similar to Theorem 4.2.43.

**CHAPTER 5**

**On ws-Homeomorphism in Topological Space**

**5.1. Introduction**

The concept of generalized homeomorphism was derived and studied in the year 1991 by Maki et all [5]. N. Nagaveni [25] introduced and investigated rwg homeomorphism in topological space. In the year 2002, M Sheik John [30] introduced and investigated w- homeomorphism in topological space. The intention of the paper is to analyze and study ws-homeomorphism, ws\*-homeomorphism and their relation with some existing homeomorphisms in topological space. Also some of their properties have been verified.

In second of this chapter, we to introduce and study ws-homeomorphism, ws\*-homeomorphism and their relation with some existing homeomorphisms in topological space. Also some of their properties have been investigated

**5.2. ws-Homeomorphism in Topological Space**

**Definition 5.2.1** A bijective map h: (P,)(Q,) is termed as ws-homeomorphism if h is both ws-continuous and ws-open.

**Example 5.2.2** Take up P=Q={1,2,3}, τ ={P, ϕ,{1},{2},{1,2}}be a topology on P and σ ={Q, ϕ,{1},{2},{1,2},{1,3}} be a topology on Q and wsC(P)= {, P, {1}, {2},{3}, {1, 3},{2, 3}}

And wsO(Q) ={Q, ϕ,{1},{2},{1,2},{1,3}}.Take up h: P→Q defined by identity map then h is ws–homeomorphism

**Theorem 5.2.3:** Each homeomorphism is a ws-homeomorphism but inverse is untrue..

Proof: Take up h: PQ be a homeomorphism. Then h is bijective, continuous and open map. Since each continuous map is ws-continuous and each open map is ws-open, h is ws-homeomorphism.

**Example 5.2.4:** Take up P=Q= {1, 2, 3}. Let = { φ ,P, {1}, {2},{1,2}}be a topology on P and ={φ, Q, {1}, {2}, {1, 2}, {1 3}} be a topology on Q and wsC(P)= {, P, {1}, {2},{3}, {1, 3},{2, 3}}. Let h: (P, )(Q, ) be a function defined by identity map is ws- homeomorphism but not a homeomorphism as the closed set H={2} in Q, (H)={2} is not a closed in P hence not a continuous function therefore not a homeomorphism.

**Theorem 5.2.5:**  If a map h: PQ is homeomorphism then the following holds.

* 1. Each semi-homeomorphism is ws-homeomorphism but inverse is untrue..
  2. Each -homeomorphism is ws-homeomorphism but inverse is untrue..
  3. Each g#-homeomorphism is ws-homeomorphism but inverse is untrue..
  4. Each \*g-homeomorphism is ws-homeomorphism but inverse is untrue..
  5. Each g\*-homeomorphism is ws-homeomorphism but inverse is untrue..
  6. Each gp-homeomorphism is ws-homeomorphism but inverse is untrue..
  7. Each -homeomorphism is ws-homeomorphism but inverse is untrue.
  8. Each regular-homeomorphism is ws-homeomorphism but inverse is untrue
  9. Each rb-homeomorphism is ws-homeomorphism but inverse is untrue.

Proof: Take up h: PQ be a homeomorphism. Then h is bijective, continuous and open map. Since each semi (respectively, , g#, \*g, g\*, gp, , regular and ) continuous map is ws-continuous and each semi-(respectively, , g, \*g , g\*, gp , regular and ) -open map is ws-open, h is ws-homeomorphism.

**Example 5.2.6:** In Example 5.2**.**3, h is ws-homeomorphism but not a semi (respectively, , g#, \*g, g\*, gp, , regular and ) homeomorphism as the closed set H={2} in Q, (H)={ 2} is not a semi (respectively, , g#, \*g, g\*, gp, ,regular and )closed in P.

**Theorem 5.2.7:** Each g#s - homeomorphism is ws-homeomorphism but inverse is untrue.

Proof: Take up h: (P, ) (Q, ) be a homeomorphism. Then h is bijective, continuous and open map. Since each g#s - continuous map is ws-continuous and each g#s- open map is ws-open, h is ws-homeomorphism.

**Example 5.2.8:** Take up P=Q= {1, 2, 3, 4}. Let = {φ, P, {1}, {2}, {1, 2},{1,2,3}}be a topology on P and ={φ,Q,{1},{1,2},{1,2,3}}be a topology on Q, and wsC(P)={P,,{1},{2},{3},{4},{1,3},{1,4},{2,3},{2,4},{3,4},{1,2,4},{1,2,4},{2,3,4}}, g#sC(P)={P, φ,{2},{3},{4},{2,3},{2,4},{3,4},{2,3,4}}, wsO(Q)={Q, φ, {1},{2},{3},{1,2}, {1,3},{1,4},{2,3},{1,2,3}, {1,2,4}, {1,3,4}}, g#sO(Q)={Q, φ ,{1},{1,2},{3,4},{1,4},{1,2,3}{1,2,4},{1,3,4}} Let h: (P, )(Q, ) be a function defined by h(1)= 1, h(2)=2, h(3)=3, h(4)=2, is ws- homeomorphism but not a homeomorphism hunction as the open set H={3,4} in Q, (H)={1,2} is not a g#s-closed in P hence not a continuous function therefore not a g#s-homeomorphism.

**Theorem** **5.2.9:** If a map h: PQ is homeomorphism then

1. If h is ws- homeomorphism then it is gspr- homeomorphism but inverse is untrue.
2. If h is ws- homeomorphism then it is gsp- homeomorphism but inverse is untrue.
3. If h is ws- homeomorphism then it is rgb- homeomorphism but inverse is untrue.

Proof: Take up h: (P,)(Q, ) be ws-homeomorphism. Then h is both ws-continuous and ws-open map. This implies that h is both gspr continuous and gspr open map. Therefore h is gspr-homeomorphism.

Similarly we can prove ii) and iii).

**Example 5.2.10:** Take up P=Q= {1, 2, 3,4}. Let = {φ, P, {1}, {2}, {1, 2},{1,2,3}}be a topology on P and ={φ,Q,{1,2},{3,4}}be a topology on Q and wsC(P)=gspC(P)={P,φ,{1},{2},{3},{4},{1,3},{1,4},{2,3},{2,4},{3,4},{1,2,4},

{1,3,4},{2,3,4}, gsprC(P)= rgbC(P)={P(P)}, wsO(Q)={,Q,{1,2},{3,4}}, gsprC(Q)= rgbC(Q)= gspC(Q)={P(Q)},and Take up h: P→Q defined by h(1)=1 , h(2)= 2 , h(3)=3, h(4)=2 then h is gsp– homeomorphism(respectively, gsp- homeomorphism, rgb- homeomorphism). But not ws-homeomorphism, as open set F= {1} in P, then, (F) = {1} is not ws-open set in Q.

**Remark 5.2.11:** Following example shows that ws-homeomorphism is independent of g, w, , gp, g, swg, rwg, wg, g\*p, gw, \*\*g, swg\*, gr, w, g\*, rg, rg R\*, rgw, wgr, gprw, pgr, g\*\*, wg\*\*, mildly g-homeomorphisms.

**Example 5.2.12:** Take up P=Q= {1, 2, 3}. Let = {φ, P, {1}, {2}, {1, 2}}be a topology on P and ={φ, Q, {1},{2,3}} be a topology on Q and wsC(P)={P, φ,{1}, {2}, {3},{2,3},{1, 3}}, Let h: (P, )(Q, ) be a function defined by h(1)=1 , h(2)= 1 , h(3)=1 is ws-homeomorphism but not g, w, , gp, g, swg, rwg, wg, g\*p, gw, \*\*g, swg\*, gr, w, g\*, rg, rg,R\*, rgw, wgr, gprw, pgr, g\*\*, wg\*\*, mildly g -homeomorphisms as the closed set H={1} in Q, (H)={1} is not a g(respectively w, , gp, g, swg, rwg, wg, g\*p, gw,\*\*g, swg\*, gr, w, g\*, rg, rg, R\*, rgw, wgr, gprw, pgr, g\*\*,wg\*\*, mildly g)closed set in P.

**Example 5.2.13:** Take up P=Q= {1, 2, 3}. Let = {φ, P, {1}, {2,3}}be a topology on P and ={φ, Q, {1},{2}, {1,2}} be a topology on Q and wsC(P)= {φ, P,{1},{2,3}}. Let h: (P, )(Q, ) be a function defined by h(1)=3 , h(2)= 2, h(3)=3, is g, w, , gp, g, swg, rwg, wg, g\*p, gw, \*\*g, swg\*, gr, w, g\*, rg, rg, R\*, rgw, wgr, gprw, pgr, g\*\*, wg\*\*, mildly g—homeomorphisms but not ws-homeomorphism as the closed set H={3} in Q, (H)={1,3} is not a ws-closed in P.

**Theorem 5.2.14:** Take up h: PQ be bijective and ws-continuous map. Henceforth following results are one and the same.

1. h is a ws-open map
2. h is a ws-homeomorphism
3. h is a ws-closed map

Proof:

(1) (2)

Take up h: PQ be bijective, ws-continuous map and ws-open map. Therefore h is a ws-homeomorphism.

(2) (3)

Take up h: PQ be a ws-homeomorphism. Let G be any closed set P and hence Gc is closed in P. Then h(Gc) is open in Q. But h(Gc)=h(G)c. Therefore h is a ws-closed map.

(3) (1)

Take up h: PQ be bijective, ws-continuous and ws-closed map. Assume G be any open set P and hence Gc is closed P. Then h(Gc) is closed Q. But h(Gc)=h(G)c. Therefore h is a ws-open map.

**Remark 5.2.15:** The composition of two ws-homeomorphism is not a ws-homeomorphism in general as seen in example 5.2.16.

**Example 5.2.16:** Take up P=Q=R= {1, 2, 3}. Let = {P, ϕ, {1}, {2}, {1,2}, {1,3}} be a topology on P, = {Q, ϕ, {1}, {2}, {1,2}} be a topology on Q and = {Z, ϕ, {1}, {2,3}} be a topology on R and wsc(P)= {P, ϕ, {2}, {3}, {1, 3}, {2, 3}} , wsc(Q)= {Q, ϕ, {1},{2}, {3}, {1, 3}, {2, 3}} wsc(R)= {R, ϕ, {1}, {2, 3}}. Define h: (P,) (Q,) by h (1) =1, h (2) =2, h(3) =3 and g: (Q,)(R, ) be a function defined by h(1)=1 , h(2)= 1, h(3)=1. Both h and g are ws-homeomorphism but their composition goh: (P,)(R, ) defined by identity function is not a ws-homeomorphism as the closed set H={1} in (R, ), but (H)={1} is not ws-closed in P.

**Definition 5.2.17:** A bijective map h: PQ is called a ws\*-homeomorphism if h and are ws-irresolute.Where ws\*-h(P, ) denote the set of all ws-homeomorphisms of P on to itself.

**Theorem 5.2.18:** Each ws\*-homeomorphism is a ws-homeomorphism but inverse is untrue.

Proof: Take up h: PQ be a ws\*-homeomorphism. Then h is bijective, h and are ws-irresolute. Since each ws-irresolute map is ws-continuous then h and are ws-continuous. Also : Q P is a ws-continuous map is one and the same. to ws-open map. Thus h is bijective, ws-continuous and ws-open map. Hence h is a ws-homeomorphism.

**Example 5.2.19:** Take up P=Q= {1, 2, 3}. Let = {φ, P, {1}, {2}, {1,2},{1,3}}be a topology on P and ={φ, Q, {1}, {2},{1,2}} be a topology on Q and wsC(P)= {φ, P, {1},{3}, {1, 3},{2,3}}, SCL(Q) ={φ, Q, {1},{2},{3},{2, 3},{1,3}}, wsC(Q)= {φ, Q, {1},{2},{3}, {1, 3},{2,3}} Let h: (P, )(Q, ) be a function defined by identity function is ws-continuous but not a ws\*-continuous function as the semi-closed set F={1} in Q, (F)={1} not a ws-closed in P.

**Theorem 5.2.20:** Each ws\*-homeomorphism is a gspr-homeomorphism (resp gsp-homeomorphism, rgb-homeomorphism) but inverse is untrue.

Proof: Proof is obtained from the previous results, definations and evidence each ws\*-homeomorphism is a ws-homeomorphism and each ws-homeomorphism is a gspr-homeomorphism (resp gsp-homeomorphism, rgb-homeomorphism)

**Example 5.2.21:** Take up P=Q= {1, 2, 3, 4}. Let = {φ, P, {1}, {2}, {1, 2},{1,2,3}}be a topology on P and ={φ,Q,{1,2},{3,4}} be a topology on Q and wsC(P)= gspC(P)={P, φ, {1},{2},{3},{4},{1,3},{1,4},{2,3},{2,4},{3,4},{1,2,4},{1,3,4},{2,3,4}}, gsprC(P)=rgbC(P)={P(P)},wsO(Q)={ φ,Q,{1,2},{3,4}},

gsprC(Q)=rgbC(Q)=gspC(Q)={P(Q)}. Take h: P→Q defined by h(1)=1 , h(2)= 2 , h(3)=3, h(4)=2 then h is gsp– homeomorphism(Respectively, gsp-homeomorphism, rgb- homeomorphism). But not ws-homeomorphism, as open set H= {1} in P, then,

(H) = {1} which is not ws-open set in Q.

**Remark 5.2.22:** From the above discussions and known facts, the relation between ws-homeomorphism with some existing homeomorphisms in topological space is shown in following figure.

g-Homeomorphism, w-Homeomorphism, - Homeomorphism, gp- Homeomorphism, g- Homeomorphism, swg- Homeomorphism, rwg-Homeomorphism, wg-Homeomorphism, g\*p-Homeomorphism, gw- Homeomorphism, \*\*g-Homeomorphism, swg\*-Homeomorphism, gr-Homeomorphism, w-Homeomorphism, g\*-Homeomorphism, rg-Homeomorphism, rg- Homeomorphism, R\*-Homeomorphism, rgw- Homeomorphism, wgr- Homeomorphism, gprw- Homeomorphism, pgr- Homeomorphism, g\*\*-Homeomorphism, wg\*\*-Homeomorphism, mildly g-Homeomorphism

Regular -Homeomorphism

Homeomorphismm

Semi-Homeomorphism

-Homeomorphism

-Homeomorphism

gp-Homeomorphism

\*g-Homeomorphism

gspr-Homeomorphism

gsp-Homeomorphism

rgb- Homeomorphism

ws- Homeomorphism

g\*-Homeomorphism

CONTINUOS

rb-Homeomorphism

Note: A B Means A implies B, but inverse is untrue.

A B Means A and B are independent.

**Theorem 5.2.23:** The composition of two ws\*-homeomorphisms is also a ws\*-homeomorphism.

Proof: Take up h: P Q and g: QR be two ws\*-homeomorphisms. Now we have to prove goh: QR be ws\*-homeomorphism. Let N be any closed set in R. Seeing that g is ws-irresolute, g(N) is ws-closed Q. Also h is ws-irresolute, ((N)) is ws-closed in P. But

((N))=(N). Therefore goh is ws-irresolute. Take W be any closed set in P. Since h is ws-irresolute, h(W) is ws-closed in Q and also g(h(W)) is ws-closed in R. But (goh)(W)=g(h(W)). Therefore (goh) is ws-irresolute. Hence (goh) is ws\*-homeomorphism.

**Theorem 5.2.24:** Under the composition of maps, the set ws\*-h(P, ) is a group.

Proof: Take up \* be a binary operation, \*: ws\*h(P, ) ws\*h(P, ) defined by h \* g = goh for all h, g ws\*h(P, ). We have, goh ws\*h(P,). Also the associative property holds in composition of maps. The identity map I : (P, ) (P, ) which ws\*h(P, ), acts as the identity element. For the inverse element, if h ws\*h(P,) then ws\*h(P,) ∋: = =I. Therefore, under the composition of maps, the set ws\*-h(P,) is a group.

**CHAPTER -6**

**WS - LOCALLY CLOSED SETS IN TOPOLOGICAL SPACES**

**6.1 Introduction.** LC-continuity and LC-irresoluteness

Bourbaki [17] defined locally closed set in (P,) as intersection of closed and open sets. Stone [87] used a new term as FG for locally closed. Further LC-continuity and LC-irresoluteness developed by Ganster and Reilly [37] . Sundaram [89] developed the knowledge of generalized locally closed sets, the class of GLC-continuous functions, GLC-irresolute functions and analysed some of their topological properties. Arockiarani [4] popularized regular generalized locally closed sets and obtained six new classes of generalized continuities using the theory of regular generalized closed sets. Recently Sheik John [83] introduced three new classes of sets denoted by w-LC(P, τ), w-LC\*(P, τ) and w-LC\*\*(P, τ) each of which contains LC(P, τ). Also various authors like Park [75], Balachandran [11] and M.K.R.S.Veera Kumar [92] have put stremendous effort to the growth of locally closed sets and locally continuous maps.

In the second section of this chapter, we describe three weaker forms of locally closed sets called ws-lc set, ws-lc\* set and ws-lc\*\* set each of which is weaker than locally closed set and examine some properties of ws-lc set, ws-lc\* set and ws-lc\*\* set.

In third section of this chapter, we introduce wsLC-continuous, wsLC\*-continuous and wsLC\*\*-continuous functions Also we introduce ws-submaximal spaces and obtain some of their properties.

In fourth section of this chapter, we define wsLC-irresolute, wsLC\*-irresolute and wsLC\*\*-irresolute functions and analyse some of their related properties.

**6.2 ws-Locally Closed Sets**

**Definition 6.2.1**: A subset D of a topological space (P, ) is called a weakly semi locally closed set (briefly wslc-set) if D = S F where S is ws-open and F is ws-closed.

The set of all weakly semi locally closed set in (P,) is denoted by wsLC(P, ).

**Example 6.2.2**: Take up P = {1,2,3,4} and τ = {φ, {1}, {2}, {1,2},{1,2,3}} be a topology on P. Then wsC(P) = {φ, P,{1}, {2}, {3}, {4}, {1,3}, {1,4} {3,4}, {2,3}, {2,4}, {1,2, 4},{1,3,4},{2,3,4}}, WSO(P) = {φ, P, {1}, {2}, {3},{1,2}, {3,4}, {1,4}, {2,3}, {2,4}, {1,2, 3}{1,2,4},{1,3,4},{2,3,4}} and wsLC(P) = {φ, P ,{1}, {2},{3},{4},{1,2},{1,3},{1,4},{1,3},{2,4},{3,4},{1,2,3},{1,2,4},{1,3,4},{2,3,4}}

**Remark. 6.2.3:**

1) A subset D of a topological space (P, τ) is a ws-lc set if its complement is the union of ws-open set and ws-closed set.

2) Each ws-open set of (P, τ) s a wslc set

**Definition 6.2.4:** A subset D of a space P is termed as wsLC\*-set if ws-open set S and a closed set F of P, D = S F.

**Definition 6.2.5:** A subset D of a space P is termed as wsLC\*\*-set if an open set S and ws-closed set F of P D = SF.

**Theorem 6.2.6**: If a subset D of(P,)is locally closed then it is wsLC(P,), wsLC\*(P,) and wsLC\*\*(P,) but inverse is untrue.

**Proof:** Take up D = U V, U is open and V is closed in (P, ). Since eachopen set is ws-open and each closed set is ws-closed Definition 6.2.3: A subset D of a topological space (P,) is said to be wsLC\*\*-set if an open set S and a ws-closed set F of (P,) D = SF., Therefore D is wsLC (P,), wsLC\*(P,) and wsLC\*\*(P,).

We use example 6.2.7 to prove the inverse of theorem is untrue.

**Example 6.2.7**: P = {1, 2, 3, 4} and τ = {φ, P, {1}, {2}, {1, 2}, {1,2, 3}} be a topology on P. Then wsC(P) = {φ, P, {1}, {2}, {3}, {4}, {1,3}, {1,4},{3,4},{2,3}, {2,4}, {1,2, 4},{1,3,4},{2,3,4}}, wsO(P) = {φ, P, {1}, {2}, {3},{1,2},{1,3} {3,4}, {1,4}, {2,3}, {2,4}, {1,2,3},{1,2,4},{1,3,4},{2,3,4}} and wsLC(P)= WSLC\*(P, τ) = {φ, P ,{1}, {2}, {3},{4},{1.2},{1,3}, {1,4}, {1,3}, {2,4},{3,4}, {1,2,3},{1,2,4},{1,3,4},{2,3,4}}, and WSLC\*\*(P, τ)= {φ, P,{1}, {2}, {3},{4},{1,2},{1,3}, {1,4}, {2,3}, {2,4},{3,4}, {1,2,3},{1,3,4},{2,3,4}} and locally closed sets or LC = { φ, P, {1}, {2}, {4}, {1,2}, {3, 4} {1,2,3},{1,3,4},{2,3,4}}, Here {3} is WSLC (P, τ), WSLC\*(P, τ), WSLC\*\*(P, τ) but not LC-set

**Theorem 6.2.8:** If a subset D of (P, τ) is WSLC\*-set then it is WSLC-set but inverse is untrue.

**Proof:** Take up D be a WSLC\*-set. Assume M as a ws-open set in P and N as closed set in P . Seeing that D is wsLC\*-set by definition, D = M n N. Since each closed set is ws-closed. Henceforth D is WSLC-set.

We use example 6.2.9 to prove the inverse of theorem is untrue.

**Example 6.2.9** Take P = {1, 2, 3, 4} and τ = {φ, {1}, {1, 2}, {1, 2, 3}} be a topology on P. Then wsC(P) = {φ, P,{1}, {2}, {3}, {4}, {1, 3}, {2, 4} {3,4}, {2,3}, {2,4},{1,2, 3},{1,3,4},{1,3,4}} wsO(P) = {φ, P, {1}, {2}, {3},{1,2}, {3,4}, {1,4}, {2,3}, {2,4}, {1,2, 3}, {1,2,4},{1,3,4},{2,3,4}} and wsLC(P)= {φ, P ,{1}, {2}, {3},{4},{1,2},{1,3}, {1,4}, {2,3}, {2,4},{3,4}, {1,2,3}{1,2,4},{1,3,4},{2,3,4}} And wsLC\*(P)= {φ, P ,{1}, {2}, {3}, {4}, {1,2},{1,3}, {1,4}, {2,3}, {3,4}, {1,2,3},{1,2,4},{1,3,4},{2,3,4}}, Then the subset {2,4} WSLC(P, ) but {2,4} WSLC\*(P, ).

**Theorem 6.2.10**: Each WSLC\*\* (P, ) is WSLC (P,) but inverse is untrue

**Proof:** Proof of above theorem follows from definitions 6.2.1 and 6.2.3.

We use example 6.2.11 to prove the inverse of theorem is untrue.

**Example 6.2.11**: Take up P = {1, 2, 3, 4} and τ = {φ, {1}, {2}, {1, 2}, {1, 2, 3}} be a topology on P. Then wsC(P) = {φ, P,{1}, {2}, {3}, {4}, {1, 3}, {1, 4} {3,4}, {2,3}, {2,4}, {1,2, 4},{1,3,4},{2,3,4}}, wsO(P) = {φ, P, {1}, {2}, {3},{1,2}, {3,4}, {1,4}, {2,3}, {2,4}, {1,2, 3},{1,2,4},{1,3,4},{2,3,4}} and wsLC(P)= WSLC\*(P, τ) ={φ, P ,{1}, {2}, {3},{4},{1,2},{1,3}, {1,4}, {2,3}, {2,4},{3,4}, {1,2,3},{1,2,4},{1,3,4},{2,3,4}}, and WSLC\*\*(P, τ)= {φ, P ,{1}, {2}, {3},{4},{1,2},{1,3}, {1,4}, {2,3}, {2,4},{3,4}, {1,2,3},{1,3,4},{2,3,4}} Here {1,2,4} WSLC (P, τ), but {1, 2,4} WSLC\* \*(P, τ).

**Theorem 6.2.12**: Each locally closed set is a wslc set but inverse is untrue.

Proof. Proof is obtained from the facts that each open set is ws-open and each closed set is ws-closed.

**Example 6.2.13**: P = {1, 2, 3, 4} and τ = {φ, {1}, {2}, {1, 2}, {1,2,3}} be a topology on P. Then wsC(P) ={φ, P, {1}, {2}, {3}, {4}, {1,3}, {1, 4} {3,4}, {2,3}, {2,4}, {1,2, 3},{1,3,4},{2,3,4}} wsO(P) = {φ, P, {1}, {2}, {3},{1,2}, {3,4}, {2,4}, {2,3}, {2,4}, {1,2,3}, {1,2,4},{1,3,4},{2,3,4}} and wsLC(P)= {φ, P ,{1}, {2}, {3},{4},{1,2},{1,3}, {1,4}, {2,3}, {2,4},{3,4}, {1,2,3}, {1,2,4}, {1,3,4}, {1,3,4}}, and and locally closed sets or LC = { φ, P, {1}, {2}, {4}, {1,2}, {3, 4} {1,2,3},{1,3,4},{2,3,4}}, Here {3} is WSLC (P, τ), but not LC-set

**Theorem 6.2.14**:.Each – lc(g -lc, gp-lc, \*g-lc)set is a wslc set but the inverse is untrue.

Proof: Proof is obtained from the fact that each -( g, gp-, \*g ) closed set is ws- closed and each -(g, gp, \*g ) open set is ws-open set.

**Example 6.2.15:** P = {1, 2, 3, 4} and τ = {φ, {1}, {2}, {1, 2}, {1,2,3}} be a topology on P. Then wsC(P) = {φ, P,{1}, {2}, {3}, {4}, {1,3}, {1, 4} {3,4}, {2,3}, {2,4}, {1,2, 3},{1,3,4},{2,3,4}},wsO(P)={φ,P,{1},{2},{3},{1,2},{3,4},{2,4},{2,3},{2,4},{1,2,3}, {1,2,4},{1,3,4},{2,3,4}}and wsLC(P)= {φ, P ,{1}, {2}, {3},{4},{1,2},{1,3}, {1,4}, {2,3}, {2,4},{3,4}, {1,2,3}{1,2,4},{1,3,4},{1,3,4}}, and – lc(g -lc, gp-lc, \*g-lc) = {φ, P, {1}, {2}, {3}, {4}, {1,2}, {3, 4} {1,2,3},{1,2,3},{1,3,4},{2,3,4}}, Here {2,3} is wsLC (P, τ), but not – lc(g -lc, gp-lc, \*g-lc) set

**Theorem 6.2.16**: Each scl – lc( -lc, -lc)set is a wslc set but the inverse is untrue.

Proof. Proof is obtained from the evidence that each scl – lc( -closed, -closed) set is ws- closed and each scl – open( -open, -open)set is ws-open set.

**Example 6.2.17**: P = {1, 2, 3, 4} and τ = {φ, {1}, {2}, {1,2}, {1,2,3}} be a topology on P. Then wsC(P) = {φ, P,{1}, {2}, {3}, {4}, {1,3}, {1, 4} {3,4}, {2,3}, {2,4}, {1,2, 3},{1,3,4},{2,3,4}} wsO(P)={φ, P, {1},{2},{3},{1,2}, {3,4}, {2,4}, {2,3}, {2,4}, {1,2,3}, {1,2,4},{1,3,4},{2,3,4}} andwsLC(P)={φ,P,{1},{2},{3},{4},{1,2},{1,3},{1,4},{2,3},{2,4},{3,4},{1,2,3}{1,2,4},{1,3,4},{1,3,4}}, and scl – lc( -lc, -lc) = {φ, P,{1},{4},{1,2}{3, 4},{1,2,3},{2,3,4}}, Here {2} is wsLC (P, τ), but not scl – lc( -lc, lc)set

**Theorem 6.2.18:**

* + 1. wslc set is gsprlc set but not conversely.
    2. wslc set is gsplc set but not conversel.
    3. wslc set is rgblc set but not conversely

**Proof:** The proof follows from the two definitions and the fact that every ws-closed (resp. ws-open) set is gspr-closed (resp. gspr-open).

Similrly we can prove the result 2) and 3)

**Example 6.2.19:** Let X= {1,2, 3} and τ={X, φ, {1}, {2,3}}. Then {2} is a gsprlc(respectively, gsplc,rgblc) set but not a ws-lc set in (X, τ).

**Remark 6.2.20:** The following example shows that wlc,lc, ,pgprlc,swglc, glc ,gwlc,gslc and rwlc are independent with wslc sets.

**Example 6.2.21:** Let X= {1, 2, 3, 4} and τ={X, φ, {1}, {2}, {1, 2}, {1, 2, 3}}. Then {1, 2} is a wslc set but not a w-lc (respectively, lc, , pgprlc, swglc, glc ,gwlc,gslc and rwlc) set in (X, τ).

**Example 6.2.22:** Let X= {1,2,3, 4} and τ={X, φ, {1, 2}, {3,4}}. Then {1} is w-lc (respectively, lc,, pgprlc, swglc, glc ,gwlc,gslc and rwlc) set but not a wslc set in (X, τ).

**Remark 6.2.23:** From the above discussions and known facts we have the following implications:

gp-lc set

-lc set

lc- set

rgb-lc set

Sc-lc set

gsp-lc set

g#-lc set

gspr-lc set

\*g-lc set

g\*-lc set

w-lc set, lc-set, -set, pgpr-lc set, swg-lc set, g-lc set ,gw-lc set,gs-lc set , rw-lc set

ws-lc set

Note: A B Means A implies B, but inverse is untrue.

A B Means A and B are independent.

**Theorem 6.2.24**: In a space (P, τ), if each semi closed set is closed then semi LC(P) ⊂ wsLC(P)

**Proof:** In general, LC(P) ⊆ wsLC(P) and LC(P) ⊆ semiLC(P). Since each semi closed set is closed then semi-LC(P) ⊂ LC(P). Then semi-LC(P) = LC(P). Therefore semi-LC(P) ⊂ semiLC(P).

**Theorem 6.2.25**: The given results are one and the same.

1. D ∈ wsLC∗(P, τ)
2. D = M ∩ cl(D) for some ws open set M
3. cl(D) − D is ws closed
4. D ∪ cl(D)c is ws open, where D is subset of a topological space (P, τ)

Proof. (1) ⇒ (2)

Take up D ∈ wsLC ∗P. Then D = M ∩ F, where M and F are ws open set and a closed set in P. Seeing that D ⊆ F, cl(D) ⊆ cl(F ) = F . Now M ∩cl(D) ⊆ (M ∩ F ) = D. That is M ∩ cl(D) ⊆ D. Also, D ⊆ M and D ⊆cl(D) then D ⊆ (M ∩ cl(D)). Thetrefore D= M ∩cl(D) for some ws open set M.

(2) ⇒ (1)

Here M is ws open and F is ws closed set. Therefore D = (M ∩ cl(D)) ∈ wsLC∗(P). (iii) ⇒ (iv)

Take up cl(D) − D is ws-closed. Now P − (cl(D) − D) = P ∩ (cl(D) − D)c = D ∪ (P − cl(D)) = D ∪ (cl(D))c is a ws- open set, That is D ∪ cl(D)c is ws open set.

(4) ⇒ (3)

Take up D ∪ cl(D)c is ws open. Now P − (D ∪ (cl(D))c) = cl(D) ∩ (P − D) = cl(D) − D is ws-closed.

(4) ⇒ (2)

Take up D ∪ cl(D)c is ws open,. Now M∩ cl(D) = (D ∪ (cl(D))c) ∪ cl(D) = (D ∩ cl(D)) ∪ ((cl(D))c ∩ cl(D)) = D ∪ φ = D

(2) ⇒ (4)

Take up D = M ∩ cl(D) for some ws open set M. Now D ∪ (cl(D)c) = (M ∩ cl(D)) ∪ (cl(D)c) = M ∩ cl(D) ∪ (cl(D))c = M ∩ P = M is a ws-open set.

**Theorem 6.2.26:** If D ∈ wsLC(P, τ) then D = M∩ wscl(D) for some open set M, where D is the subset of a topological space (P, τ)

**Proof:** Take up D ∈ wsLC(P). Then D = M ∩ F where M and F are ws closed and ws open sets in (P,) respectively. Since D ⊆ F, wscl(D) ⊆ wscl(F )=F. Now M ∩ wscl(D) ⊆ (M ∩ F ) = D. Dlso, since D ⊆ M and D ⊆ wscl(D) then D ⊆ (M ∩ wscl(D)). Therefore D = M ∩ wscl(D).

**Theorem 6.2.27:** If A ∈ wsLC∗∗(P, τ) then D = M ∩ wscl(D), where M is an open set and D is subset of a topological space (P, τ)

Proof. Take up D ∈ wsLC∗∗(P, τ). Then D = M ∩ F, where M is an open set and F is a ws- closed set. Seeing that D ⊆ F and wscl(D) ⊆ wscl(F ) = F thus M ∩ wscl(D) ⊆ (M ∩ F ) = D. That is M ∩ wscl(D) ⊆ D. Also D ⊆ M and D ⊆ wscl(D) then D ⊆ M ∩ wscl(D). Thus, D = M ∩ wscl(D) for some open set M.

**6.3 WSLC-Continuous Maps**

**Definition 6.3.1:** The map h: P → Q is called a WSLC-continuous (respectively WSLC∗-continuous and WSLC∗∗ continuous) if the inverse image of each open set in Q is wslc-set(respectively wslc∗-set and wslc∗∗ -set) in (P, τ).

**Example 6.3.2:** Take up P = Q = {1, 2, 3}, = {φ, {1}, {2}, {1, 2}} and σ = {φ, Q, {1}, {2, 3}}. Then WSLC(P) = P (P) and h: P → Q is defined as identity map is a WSLC-continuous.

**Theorem 6.3.3**: Take up h: (P,) → Qbe a function. Then

If H is locally continuous then it is WSLC-continuous but inverse is untrue

Proof: Proof is obvious from the well known definitions and evidence

**Example 6.3.4:** Take P = {1, 2, 3, 4} and τ = {φ, P {1}, {2}, {1, 2}, {1, 2, 3}} be a topology on P. = {φ, Y, {1},{ 2, 3}} Then wsC(P) = {φ, P,{1}, {2}, {3}, {4}, {1,3}, {1, 4} {3,4}, {2,3}, {2,4}, {1,2, 3},{1,3,4},{2,3,4}} wsO(P) = {φ, P, {1}, {2}, {3},{1,2}, {3,4}, {2,4}, {2,3}, {2,4}, {1,2,3}, {1,2,4},{1,3,4},{2,3,4}} and wsLC(P)= {φ, P ,{1}, {2}, {3},{4},{1,2},{1,3}, {1,4}, {2,3}, {2,4},{3,4}, {1,2,3}{1,2,4},{1,3,4},{1,3,4}}, LC(P)={ φ P, {1}, {2}, {4}, {1, 2}, {3,4}, {1,2,3},{1,3,4},{2,3,4}} and Let h: (P, )(Y, ) be a function defined by identity map is wslc-continuous but not aLC continuous function as the open set F={2,3} in Q, (F)={2,3} is not a locally continuos closed set in P.

**Theorem 6.3.5:** Each – lc continuous (g -lc continuous, gp-lc continuous, \*g-lc continuous) is a wslc- continuous but the inverse is untrue.

Proof: Proof is obvious from the well known definitions and evidence.

**Example 6.3.6:** Take P = {1, 2, 3, 4} and Q = {1, 2, 3} , τ = {φ, {1}, {2}, {1, 2}, {1, 2, 3}} be a topology on P. = {φ, Q, {1},{ 2, 3}} be a topology on Q, wsC(P) = {φ, P,{1}, {2}, {3}, {4}, {1,3}, {1, 4} {3,4}, {2,3}, {2,4}, {1,2, 3},{1,3,4},{2,3,4}}, wsO(P) = {φ, P, {1}, {2}, {3},{1,2}, {3,4}, {2,4}, {2,3}, {2,4}, {1,2,3}, {1,2,4},{1,3,4},{2,3,4}} and wsLC(P)= {φ, P ,{1}, {2}, {3},{4},{1,2},{1,3}, {1,4}, {2,3}, {2,4},{3,4}, {1,2,3}{1,2,4},{1,3,4},{1,3,4}}, and – lc(g -lc, gp-lc, \*g-lc) = { φ, P, {1}, {2}, {3}, {4}, {1,2}, {3, 4} {1,2,3},{1,2,4},{1,3,4},{2,3,4}}, Here {2,3} is wsLC (P, τ), but not – lc(g -lc, gp-lc, \*g-lc) set. Let h: (P, )(Y, ) be a function defined by identity map is wslc-continuous but not a – lc continuous (g -lc continuous, gp-lc continuous, \*g-lc continuous) )function as the open set F={2,3} in Q, (F)={2,3} is not a – lc(g -lc, gp-lc, \*g-lc) closed set in P.

**Theorem 6.3.7:** Each semilc-continuous ( -lc continuos, -lc continuos)set is a wslc continuos, but the inverse is untrue.

Proof: Proof is obvious from the well known definitions and evidence.

**Example 6.3.8:** Take P = {1,2,3,4} and Q= {1,2,3} and τ = {φ, {1},{2}, {1,2}, {1,2,3}} be a topology on P. = {φ, Q, {1},{ 2, 3}} be a topology on P. wsC(P) = {φ, P,{1}, {2}, {3}, {4}, {1,3}, {1, 4} {3,4}, {2,3}, {2,4}, {1,2, 3},{1,3,4},{2,3,4}} wsO(P) = {φ, P, {1}, {2}, {3},{1,2}, {3,4}, {2,4}, {2,3}, {2,4}, {1,2,3}, {1,2,4},{1,3,4},{2,3,4}} and wsLC(P)= {φ, P ,{1}, {2}, {3},{4},{1,2},{1,3}, {1,4}, {2,3}, {2,4},{3,4}, {1,2,3}{1,2,4},{1,3,4},{1,3,4}}, and scl – lc( -lc, -lc) = {φ, P,{1},{4},{1,2}{3, 4},{1,2,3},{2,3,4}} Let h: (P, )(Y, ) be a function defined by identity map is wslc-continuous but not a semilc-continuous( -lc continuos, -lc continuos )function as the open set F={2,3} in Q, (F)={2,3} is not a scl – lc( -lc, -lc) closed set in P.

**Theorem 6.3.9:** Every ws-lc continuous is gspr (respectively gsp -lc continuos, rgb-lc continuos) continuos, but the converse is not true.

Proof. The proof follows from the known definitions and facts.

**Example 6.3.10:** X = {1, 2, 3} and Y= {1, 2, 3} and τ = {X, φ, {1}, {2, 3}} be a topology on X. = {φ, Y, {1}, {1, 2}, {1, 2, 3}}, Then wsC(X) = {φ, X, {1}, {2, 3}}. and wsLC(X)= {φ, X {1}, {2, 3}}, and gspr – lc(gsp-lc, rgb-lc) = {P(X)} Let h: (X, )(Y, ) be a function defined by h(1)= 1, h(2)=2, h(3)=3, is gspr – lc(respectively, gsp-lc, rgb-lc) -continuous but not a ws-lc continuos function as the open set H={1,2,3} in Q, (H)={1,2,3} is not a ws – lc set in X.

**Theorem 6.3.11:** Each WSLC\*\* conntinuous is WSLC-continuous.but inverse is untrue

Proof: Proof is obvious from the well known definitions and evidence.

**Example 6.3.12:** P = {1,2,3,4} and Y= {1,2,3} and τ = {φ, {1},{2}, {1,2}, {1,2,3}} be a topology on P. = {φ, Y, {1},{ 2, 3}}, wsC(P) = {φ, P,{1}, {2}, {3}, {4}, {1,3}, {1, 4} {3,4}, {2,3}, {2,4}, {1,2, 3},{1,3,4},{2,3,4}}, wsO(P) = {φ, P, {1}, {2}, {3},{1,2}, {3,4}, {2,4}, {2,3}, {2,4}, {1,2,3}, {1,2,4},{1,3,4},{2,3,4}} and wsLC(P)= {φ, P ,{1}, {2}, {3},{4},{1,2},{1,3}, {1,4}, {2,3}, {2,4},{3,4}, {1,2,3}{1,2,4},{1,3,4},{1,3,4}} and wsLC\*\*P)= {φ, P ,{1}, {2}, {3},{4},{1,2},{1,3}, {1,4}, {2,3}, {2,4},{3,4}, {1, 2, 3},{1,3,4},{2,3,4}}, Let h: (P, )(Q, ) be a function defined by h(1)= 2, h(2)=2, h(3)=4, is wslc-continuous but not a wsLC\*\*continuous function as the open set H={2,3} in Q, (H)={1,2,3} is not a wsLC\*\* closed set in P.

**Remark 6.2.13:** From the above discussions and known facts we have the following implications:

- LC continuity

gsp- LC continuity

g\*- LC continuity

rgb- LC continuity

- LC continuity

g#- LC continuity

SC- LC continuity

gspr- LC continuity

WSLC-CONTINUITY

gp- LC continuity

\*g- LC continuity

Note: A B Means A implies B, but inverse is untrue.

A B Means A and B are independent.

**Theorem 6.3.14:** Let h: P → Q is a function defined on a door space (P,) is a WSLC-continuous (respectively WSLC∗ continuous and WSLC∗∗-continuous).

Proof. Take up h be a function, where Qbe any space and (P,) is a door space. Let E ∈ σ. Then h−1 (E) is either open or closed. In both the cases, h−1 (E) ∈ WSLC(P, ) (respectively WSLC∗(P), WSLC∗∗(P)). Therefore h is WSLC-continuous (respectively WSLC∗ continuous and WSLC∗∗-continuous).

**Theorem 6.3.15:** Take h : P → Q be WSLC-continuous and E is an open set in Q. Then h : (P, ) → (E, E) is WSLC-continuous.

**Proof.** Take up U as an open set in E and hence open in (Q, σ). Then h−1(V ) is wslc-set in (P, ), by hypothesis. Therefore h : (P, ) → (E, E) is WSLC-continuous.

**Theorem 6.3.16:** Take h: P → Q is WSLC-continuous (respectively WSLC∗-continuous and WSLC∗∗-continuous) and g: Q → R is continuous then goh : P → R is WSLC-continuous (respectively WSLC∗-continuous and WSLC∗∗-continuous).

**Proof.** Take up N as any open set in (R, η). Seeing that g is continuous map, g−1(N ) is open in (Q, σ). Also h is WSLC-continuous, h−1(g−1(N )) = (gof)−1(N ) is a wslc set in (P, ). Then goh is WSLC-continuous.

**6.4** WSLC irresolute maps and their properties

**Definition.6.4.1:** A map h: P → Q is called wslc-irresolute (respectively WSLC∗-irresolute, WSLC∗∗-irresolute) if inverse image of each ws locally closed set in Qis ws locally closed set (recpectively wslc∗ locally closed, wslc∗∗ locally closed) set in (P, ).

**Example 6.4.2:** Take up P = Q = {1,2, 3}, = {φ, {1}, {2}, {1,2}} and σ = {φ, Q, {1}, {2, 3}}. Then WSLC(P) = P (P) and h: P → Q is defined as identity map is a WSLC-irresolute map.

**Theorem. 6.4.3:** If h: (P, ) → Qis ws-irresolute then h is WSLC-irresolute but inverse is untrue.

**Proof.** Take up h : (P, ) → Qis ws-irresolute. Let K ∈ WSLC(P). Then K = L ∩ M , where L is ws-open set and M is ws-closed set in (Q, σ). Now, h−1(K ) = h−1(L ∩ M ) = h−1(L) ∩ h−1(K ). Seeing that h is ws-irresolute, h−1(L) is ws-open and h−1(M) is ws closed set in (P, ), therefore h−1(M) wslc(P), Hence the theorem.

**Example. 6.4.4:** Take up P = Q = {1, 2, 3, 4}, and = {φ, P, {1}, {1, 2}, {1, 2, 3}}. = {φ, Q, {1}, {2}, {1, 2}, {1, 2, 3}}, wsC(P) = {φ, P, {2}, {3}, {4}, {1, 4}, {2,3}, {2,3},{3,4}, {1,2, 4},{1,3,4},{2,3,4}}, wsC(Q) = {φ, Q,{1}, {2}, {3}, {4}, {1, 3}, {1, 4} {2,3}, {2,4}, {3,4}{1,2, 4},{1,3,4},{2,3,4}}. Let h : P → Q be an identity map but not a ws-irresolute map, as the inverse image of the ws-closed set {1} in Qis {1} in (P, ), which is not a ws-closed set in (P, ).

**Theorem. 6.4.5:** If a map h: (P, ) → Qis WSLC-irresolute then it is wslc-continuous.

Proof. Result is obvious from the well known definitions and evidence that each open set is a wslc-set

**Theorem 6.4.6:** Any function h : (P, τ) → Qdefined on a door space is WSLC-irresolute.

Proof. The proof is obvious.

**Theorem 6.4.7:** Consider two functions h: P → Qand g: Q→ (R, η). Then

1. goh : (P, τ) → (R, η) is WSLC-irresolute if both h and g are WSLC-irresolute maps.
2. goh: (P, τ) → (R, η) is WSLC-continuous if h is WSLC-continuous and g is WSLC-irresolute.

Proof. Take up L be any wslc set in (R, η). By hypothesis, g−1(L ) is a wslc set in (Q, σ). Also h is WSLC-irresolute, h−1(g−1(L )) = (goh)−1(L ) is a wslc set in (P, τ). Therefore goh is WSLC-irresolute.

This theorem also hold for WSLC∗-irresolute map and WSLC∗∗-irresolute map. ii) Take up L be any open set in (R, η). By hypothesis, g−1(L ) is a wslc set in (Q, σ). Also, since h is WSLC-irresolute, h−1(g−1(L )) = (goh)−1(L ) is a wslc set in (P, τ). Therefore goh is WSLC-continuous.

**Definition 6.4.8:**  Function h: P → Qis said to be sub-WSLC∗-continuous if there is a basis β for Q h−1(V ) ∈ WSLC∗(P) for eachV ∈ β.

**Theorem 6.4.9:** If h : (P, τ) → Qis WSLC∗-continuous then it is sub-WSLC∗-continuous.

**Proof.** Take up h: P → Q be WSLC∗-continuous and β is a bais for Qand u ∈ β. Since h is WSLC∗-continuous, h−1(u) ∈ WSLC∗(P , ). Therefore h is sub-WSLC∗-continuous.

**Theorem. 6.4.10:** If h: P → Q is sub-LC-continuous then then it is sub-WSLC∗-continuous.

**Proof.** Result is obvious from the well known definitions and evidence that LC(P,) ⊆ WSLC(P, ) .

**Theorem 6.4.11:** Take up D and E be two subsets of a topological space (P, τ). Then these statements are verified.

1. If D ∈ WSLC∗(P) and E ∈ WSLC∗(P) then (D ∩ E) ∈ WSLC∗(P).
2. If D ∈ WSLC∗∗(P) and E is open then (D ∩ E) ∈ WSLC∗∗(P).
3. If D ∈ WSLC(P) and E is ws-open then (D ∩ E) ∈ WSLC(P)
4. If D ∈ WSLC∗(P) and E is ws-open then (D ∩ E) ∈ WSLC∗(P).
5. If D ∈ WSLC∗(P) and E is closed then (D ∩ E) ∈ WSLC∗(P).

**Proof.** i) Take up D ∈ WSLC∗(P) and E ∈ WSLC∗(P). and also we know that ∃two ws-open sets K and L : D = K ∩ cl(D) and E = L ∩ cl(E). Now D ∩ E = (K ∩ L) ∩ (cl(D) ∩ cl(E)). Here K ∩ L is ws-open and (cl(D) ∩ cl(E)) is closed. Therefore (D ∩ E) ∈ WSLC∗(P).

ii) Take up D ∈ WSLC∗∗(P) and E is open. Then D = K ∩ G, where K is an open set and G is a ws-closed. Consider D ∩ E = (K ∩ E) ∩ G, where K ∩ E is open and G is ws-closed. Therefore (D ∩ E) ∈ WSLC∗∗(P).

iii) Take up D ∈ WSLC(P) and E is ws-open. Then D = K ∩ G, where P is ws-open and G is ws-closed. Consider D ∩ E = (K ∩ E) ∩ G where K ∩ E is ws-open and G is ws-closed. Therefore (D ∩ E) ∈ WSLC(P).

Similarly we can prove iv) and v).

**CHAPTER - 7**

**ws-CLOSED SETS AND ws-CONTINUOUS MAPS IN BITOPOLOGICAL SPACES**

**7.1 Introduction.**

The triple (P, τ1, τ2) where τ1 and τ2 are topologies and P is a set is called a bitopological space. In 1963 Kelly [45] started the investigation of such spaces. He generalized the topological concepts to bitopological setting and published a large number of papers. After Kelly’s work on bitopological spaces, various authors, like Arya and Nour [7], Di Maio and Noiri [24], Fukutake [34], Nagaveni [65], Maki, Sundaram and Balachandran [55], Sheik John [85], Sampath Kumar[82], Patty [77], Arockiarani [3], Gnanambal [40], Reilly [8], Rajamani and Viswanthan [80], and Popa [78] have made stremndous effort on all concepts of topology nto bitopological spaces instead of topological spaces.

In chapter II, we have introduced and analysed the concept of ws-closed sets, ws-open sets and ws-closure operator in topological spaces. In chapter III, we have introduced and investigated some properties of ws-continuous maps and ws-irresolute maps in topological spaces.

In section 2 of this chapter, (i,j)-ws-closed sets in bitopological space have been introduced and studied. In section 3 of this chapter, we have introduced (i, j)-ws-open sets in bitopological space and study some of their related properties. In section 4 of this chapter, we shall use the (i, j)-ws-closed subsets of bitopological space (P, τ1, τ2) to define a new closure operator “ (i, j)-ws-cl” , and thus new topology τws(i, j) on the space and shall examine some of the properties of this new topology.

In section 5 of this chapter, a new kind of maps called Dws(i, j)-σ k-continuous maps in bitopological spaces are introduced and investigated. During this process, some of their properties are obtained. Also, we have introduced the concept ofws-bi-continuity, ws-s-bi-continuity and examine some related properties.

Throughout this chapter (P, τ1, τ2), (Y, σ1, σ2) and (Z, η1, η2) denote nonempty bitopological spaces on which no separation axioms are assumed, unless otherwise mentioned and fixed integers i, j, k, e, m, n∈{1, 2}.

**7.2 (τi, τj) ws-closed sets and their basic properties.**

Here we work on concept of (τi, τj)-ws-closed sets which are developed on a bitopological space in analogy with ws-closed sets in topological spaces. From now on, τ-cl(D) denotes the closure of D relative to a topology τ.

**Definition 7.2.1:** Take up i, j ∈ {1, 2} be fixed integers. In a bitopological space (P, τ1, τ2), a subset D ⊂ P is said to be (τi, τj)-ws-closed (briefly, (i, j)-ws-closed) setif τj-scl(D) ⊂ G and G ∈ wO(P, τi).

We denote the family of all (i, j) ws-closed sets in a bitopological space (P, τ1, τ2) by Dws(τi, τj) or Dws(i, j).

**Remark 7.2.2:** By setting τ1 = τ2 in Definition 7.2.1, an (i, j)-ws-closed set reduces to a ws-closed set in P.

Firstly we express that the class of (i, j)-ws-closed sets properly lies between the class of (i, j)-g#-closed sets and the class of (i, j)-gspr-closed sets.

**Theorem 7.2.3:** If M is (i, j) -g#-closed subset of (P, τ1, τ2), then M is (i, j)-w-closed.

**Proof:** Take up M be a (i, j)-ws-closed subset of (P, τ1, τ2). Assume G∈WO(P, τi) be ∋: A⊂ G. Seeing that WO(P, τi) ⊂ SO(P, τi), hence we have G∈SO(P, τi). Then by hypothesis, τj-scl(M) ⊂ G. Henceforth M is (i, j)-g#-closed.

We use example 7.2.4 to prove the inverse of theorem is untrue.

**Example 7.2.4:** Take up P= {1,2,3}, τ1={P, φ, {1}, {2,3}} and τ2={ P, φ, {1}, {2}, {1, 2}}. Then the subsets {1} and {2} are (1, 2)-ws-closed sets, but not (1, 2)-g#-closed sets in the bitopological space (P, τ1, τ2).

**Theorem 7.2.5:** If A is (i, j) -\*g-closed subset of (X, τ1, τ2), then A is (i, j)-w-closed.

**Proof:** Let A be a (i, j)-ws-closed subset of (X, τ1, τ2). Let G∈WO(X, τi) be such that A ⊂ G. Since WO(X, τi) ⊂ SO(X, τi), we have G∈SO(X, τi). Then by hypothesis, τj-scl(A) ⊂ G. Henceforth A is (i, j)-g#-closed.

We use example 7.2.6 to prove the inverse of theorem is untrue.

**Example 7.2.6:** Let X= {1, 2, 3}, τ1={X, φ, {1}, {2,3}} and τ2={ X, φ, {1}, {2}, {1, 2}}. Then the subset {2} is (1, 2)-ws-closed sets, but not (1, 2)-g#-closed sets in the bitopological space (X, τ1, τ2).

**Theorem 7.2.7:** If M is a (i, j)-g\*-closed subset of (P, τ1, τ2), then M is (i, j)-ws-closed.

**Proof:** Take up M be a (i, j)-ws-closed subset of (P, τ1, τ2). Assume G∈WO(P, τi) be ∋: M ⊂ G. Seeing that WO(P, τi) ⊂ SO(P, τi), hence we have G∈SO(P, τi). Then by hypothesis, τj-scl(M) ⊂ G. Henceforth M is a (i, j)- g\*-closed.

We use example 7.2.8 to prove the inverse of theorem is untrue.

**Example 7.2.8:** Take up P= {1,2,3}, τ1={P, φ, {1}, {2,3}} and τ2={ P, φ, {1}, {2}, {1, 2}}. Then the subset {2} is (1, 2)-ws-closed sets, but not (1, 2)- g\*-closed sets in the bitopological space (P, τ1, τ2).

**Theorem 7.2.9:** If M is a (i, j)--closed subset of (P, τ1, τ2), then M is (i, j)-ws-closed.

**Proof:** Take up M be a (i, j)-ws-closed subset of (P, τ1, τ2). Assume G∈WO(P, τi) be ∋: M ⊂ G. Seeing that WO(P, τi) ⊂ SO(P, τi), hence we have G∈SO(P, τi). Then by hypothesis, τj-scl(M) ⊂ G. Henceforth M is a (i, j)- -closed.

We use example 7.2.10 o prove the inverse of theorem is untrue.

**Example 7.2.10:** Take up P= {1,2,3}, τ1={P, φ, {1}, {2,3}} and τ2={ P, φ, {1}, {2}, {1, 2}}. Then the subset {2} is (1, 2)-ws-closed sets, but not (1, 2) - -closed sets in the bitopological space (P, τ1, τ2).

**Theorem 7.2.11:** If M is a (i, j)--closed subset of (P, τ1, τ2), then M is (i, j)-ws-closed.

**Proof:** Take up A be a (i, j)-ws-closed subset of (P, τ1, τ2). Assume G∈WO(P, τi) be ∋: M ⊂ G. Seeing that WO(P, τi) ⊂ SO(P, τi), hence we have G∈SO(P, τi). Then by hypothesis, τj-scl(D) ⊂ G. Henceforth D is a (i, j)- -closed.

We use example 7.2.12 to prove the inverse of theorem is untrue.

**Example 7.2.12:** Take up P= {1,2,3}, τ1={P, φ, {1}, {2, 3}} and τ2={ P, φ, {1}, {2}, {1, 2}}. Then the subset {2} is (1, 2)-ws-closed sets, but not (1, 2)- -closed sets in the bitopological space (P, τ1, τ2).

**Theorem 7.2.13:** If D is a (i, j)-rb -closed subset of (P, τ1, τ2), then D is (i, j)-ws-closed.

**Proof:** Take up D be a (i, j)-ws-closed subset of (P, τ1, τ2). Assume G∈WO(P, τi) be ∋: M ⊂ G. Seeing that WO(P, τi) ⊂ SO(P, τi), hence we have G∈SO(P, τi). Then by hypothesis, τj-scl(M) ⊂ G. Henceforth M is a (i, j)- rb-closed.

We use example 7.2.14 to prove the inverse of theorem is untrue.

**Example 7.2.14:** Take up P= {1,2,3}, τ1={P, φ, {1}, {2, 3}} and τ2={ P, φ, {1}, {2}, {1, 2}}. Then the subset {2} is (1, 2)-ws-closed sets, but not (1, 2)- rb -closed sets in the bitopological space (P, τ1, τ2).

**Theorem 7.2.15:** If M is a (i, j)-ws-closed subset of (P, τ1, τ2), then M is (i, j)-gspr-closed.

**Proof:** Take up M be a (i, j)-ws-closed subset of (P, τ1, τ2). Assume G∈WO(P, τi) be ∋: M ⊂ G. Since WO(P, τi) ⊂ SO(P, τi), hence we have G∈SO(P, τi). Then by hypothesis, τj-Scl(M) ⊂ G. Also τj-spcl(M) ⊂ τj-Scl(M) which implies τj-spcl(M) ⊂ G. Therefore M is (i, j)-gspr-closed.

We use example 7.2.16 to prove the inverse of theorem is untrue.

**Example 7.2.16:** Take up P={1, 2, 3}, τ1={P, φ, {1}, { 2,3}} and τ2={ P, φ, {1}, {2},{1,2}}. Then the subsets {1,2} are (1, 2)-gspr-closed sets but not (1, 2)-ws-closed sets in the bitopological space (P, τ1, τ2).

**Theorem 7.2.17:** If M is a (i, j)-ws-closed subset of (P, τ1, τ2), then M is (i, j)-gsp-closed.

**Proof:** Take up M be a (i, j)- ws-closed subset of (P, τ1, τ2). Let G∈WO(P, τi) be ∋: M ⊂ G. Since WO(P, τi) ⊂ SO(P, τi), we have G∈SO(P, τi). Then by hypothesis, τj-scl(A) ⊂ G. Also τj-spcl(M) ⊂ τj-scl(M) which implies τj-spcl(M) ⊂ G. Henceforth M is (i, j)-gsp-closed.

We use example 7.2.18 to prove the inverse of theorem is untrue.

**Example 7.2.18:** Take up P={1,2,3}, τ1={P, φ, {1}, {2, 3}} and τ2={ P, φ, {1}, {2},{1,2}}. Then the subsets {1,2} are (1, 2)-gsp-closed sets but not (1, 2)-ws-closed sets in the bitopological space (P, τ1, τ2).

**Theorem 7.2.19:** If M is a (i, j)-ws-closed subset of (P, τ1, τ2), then M is (i, j)-rgb-closed.

**Proof:** Take up M be a (i, j)-ws-closed subset of (P, τ1, τ2). Assume G∈WO(P, τi) be ∋: M ⊂ G. Seeing that WO(P, τi) ⊂ SO(P, τi), hence we have G∈SO(P, τi). Then by hypothesis, τj-scl(M) ⊂ G. Also τj-bcl(M) ⊂ τj-scl(M) which implies τj-bcl(M) ⊂ G. Therefore M is (i, j)-rgb-closed.

We use example 7.2.20 to prove the inverse of theorem is untrue.

**Example 7.2.20:** Take up P={1,2,3}, τ1={P, φ, {1}, { 2,3}} and τ2={ P, φ, {1}, {2},{1,2}}. Then the subsets {1, 2} are (1, 2)-rgb-closed sets but not (1, 2)-ws-closed sets in the bitopological space (P, τ1, τ2).

**Remark 7.2.21:** τj-rw-closed(R\*-closed)sets, and (i,j)-ws-closed sets are independent as seen from the following examples.

**Example 7.2.22:** Take up P= {1,2,3}, τ1={P, φ, {1}, {2,3}} and τ2={P, φ, {1},{2},{1, 2}}. Then the subset {2} is (1, 2)- ws-closed sets, but not τ2- rw-closed(R\*-closed)sets set in the bitopological space (P, τ1, τ2).

**Example 7.2.23:** Take up P={1,2,3}, τ1={P, φ, {1}, {2}, {1, 2}} and τ2={P, φ, {1}, {2},{1,2},{1, 3}}. Then the subsets {1,2} are τ2- rw-closed(R\*-closed)sets but not (1, 2)-ws-closedrsets in the bitopological space (P, τ1, τ2).

**Remark 7.2.24:** From the above discussions and known facts we have the following implications:

(i,j )-gsp bitopological spaces

(i,j )-g\*- bitopological spaces

(i,j )-rgb- bitopological spaces

(i,j )-rw- bitopological spaces

(i,j )-- bitopological spaces

(i,j )-g#- bitopological spaces

(i,j )-gspr- bitopological spaces

(i,j )-ws-bitopological spaces

(i,j )-gp-bitopological spaces

(i,j )-\*g- bitopological spaces

(i,j )-R\*- bitopological spaces

Note: A B Means A implies B, but inverse is untrue.

A B Means A and B are independent.

**Theorem 7.2.24:** If M, N∈ Dws(i, j), then MN∈ Drw(i, j).

**Proof:** Take up G∈ WO(P, τi) be ∋: MN ⊂ G. Then M ⊂ G and N ⊂ G. Seeing that M, N∈ Dws(i, j), we have τj-scl(M) ⊂ G and τj-scl(N) ⊂ G. That is τj-scl(M) ∪ τj-scl(N) ⊂ G. Also τj-scl(M) τj-scl(N)= τj-cl(MN) and so τj-scl(MN) ⊂ G. Therefore MN∈ Dws(i, j).

**Remark 7.2.25:** The union of two (i, j)-ws-closed sets is generally not a (i, j)-ws-closed set as seen from example given below.  
 **Example 7. 2.26:** Take up P={1,2,3}, τ1={P, φ, {1},{2, 3}} and τ2={ P, φ, {1}, {2}, {1, 2}}. Then subsets {1} and {2} are (1, 2)-ws-closed sets, but {1}{2}={1,2} is not a (1, 2)-ws-closed set in the bitopological space (P, τ1, τ2).

**Remark 7.2.27:** As usual, The family Dws(1, 2) is unequal to the family Dws(2, 1) as observed in the next example.

**Example 7.2.28:** Take up P= {1,2,3}, τ1={P, φ, {1}, {2}, {1, 2}} and τ2={P, φ, {3}, {2, 3}}. Then Dws(1, 2)={P, φ, {1}, {3},{2,3},{1,3}} and Dws(2, 1)={P, φ, {1}, {2}, {3},{2, 3}, {1, 3}}. Therefore Dws(1, 2)≠Dws(2, 1).

**Theorem 7.2.29:** If τ1 ⊂ τ2 and SWO(P, τ1)⊂ WO(P, τ2) in (P, τ1, τ2), then Dws(τ1, τ2) ⊃ Dws(τ2, τ1).

**Proof:** Take up M∈ Dws(τ2, τ1). That is M is a (2, 1)-ws-closed set. To prove that M∈ Dws(τ1, τ2). Let G∈WO(P, τ1) be ∋: M ⊂ G. Since WO(P, τ1)⊂ WO(P, τ2), we have G∈ WO(P, τ2). As M is a (2, 1)-ws-closed set, we have τ1-scl(M) ⊂ G. Since τ1 ⊂ τ2, we have τ2-scl(M) ⊂ τ1-scl(M) and it follows that τ2-scl(M) ⊂ G. Hence M is (1, 2)-ws-closed. That is M ∈ Dws(τ1, τ2). Henceforth Dws(τ1, τ2) ⊃ Dws(τ2, τ1).

**Theorem7.2.30:** Take up i, j be fixed integers of {1, 2}. For each P of (P, τ1, τ2), {P} is a w-open in (P, τi) or {P}c is (i, j)-ws-closed.

**Proof:** Pretend {P} is not w-open in (P, τi). Then {P}c is not w-open in (P, τi). Now w-open in (P, τi) containing {P}c is P alone. Also τj-scl({P}c) ⊂ P. Hence {P}c is (i, j)-ws-closed.

**Corollary 7.2.31:** If M is (i, j)-ws-closed in (P, τ1, τ2), then M is τj-closed iff τj-scl(M)−M is a τi-w-open set.

**Proof:** Pretend M is τj-closed. Thus τj-scl(M) = M and so τj-scl(M)−M =φ, is τi-regular semiopen set.

Inversely, Pretend τj-scl(M)−M is a τi-w-open. Since M is (i, j)-ws-closed, seeing that τj-scl(M)−M non-empty τi-w-open set. Henceforth τj-scl(M)−M=φ. That is τj-scl(M)=M and hence M is τj-closed.

**Theorem 7.2.32:** In a bitopological space(P, τ1, τ2), WO(P, τi) ⊂ {F⊂P: Fc∈τj} iff each subset of (P, τ1, τ2) is a (i, j)-ws-closed .

**Proof:** Pretend that WO(P, τi) ⊂ {F⊂P: Fc∈τj}. Let M be any subset of P. Let G∈ WO(P, τi) be M ⊂ G. Thus τj-scl(G)=G. also τj-scl(M) ⊂ τj-scl(G)=G. That is τj-scl(M) ⊂ G. henceforth M is a (i, j)-ws-closed set.

Inversely, Pretend that each subset of (P, τ1, τ2) is a (i, j)-ws-closed set. Let G∈ WO(P, τi). Since G ⊂ G and G is (i, j)-ws-closed, we have τj-scl(G) ⊂ G. Thus τj-scl(G) =G and so G is τj-closed. That is G∈{F⊂P: Fc∈τj}. Henceforth WO(P, τi) ⊂ {F⊂P: Fc∈τj}.

**Theorem 7.2.33:** Pretend M be a (i, j)-ws-closed subset of a bitopological space (P, τ1, τ2). If M is τi-w-open, then M is τj-closed.

**Proof:** Take up M be τi-w-open. Now M⊂M. Then by hypothesis τj-scl(M) ⊂ M. Therefore τj-scl(M) = M. That is M is τj-closed.

**Theorem 7.2.34:** If M is a (i, j)-ws-closed set and τi ⊂ WO(P, τi), then τj-scl({P}) ∩ M ≠φ for each P ∈ τj-scl(M).

**Proof:** Take up M be (i, j)-ws-closed and τi ⊂ WO(P, τi). Suppose τi-scl({P}) ∩ M = φ for some P ∈ τj-scl(M), then M ⊂ (τi-scl({P}))c. Now (τi-scl({P}))c∈τi⊂ WO(P, τi), by hypothesis. That is (τi-scl({P}))c is τi-regular semiopen. Since M is (i, j)-ws-closed, Hence we have τj-scl(M) ⊂ (τi-scl({P}))c. This shows that P ∉ τj-scl(M). This contradicts the assumption.

**Theorem 7.2.35:** If M is a (i, j)-ws-closed set and M ⊂ N ⊂ τj-scl(M), then N is (i, j)-ws-closed.

**Proof:** Take up G be a τi-w-open set N ⊂ G. As M is a (i, j)-ws-closed set and M ⊂ G, we have τj-scl(M) ⊂ G. Now N ⊂ τj-scl(M) which implies, τj-scl(N) ⊂ τj-scl{τj-scl(M)}= τj-scl(M) ⊂ G. Thus τj-scl(N) ⊂ G. Therefore N is a (i, j)-ws-closed set.

**Corollary 7.2.36**: Let K and L be subsets of space (P, τ1, τ2). If L is (i, j)-ws-open and K ⊃ τj-int(L), then K∩L is (i, j)-ws-open.

**Proof:** Take up L be (i, j)-ws-open and K ⊃τj-int(L).That is τj-int(L)⊂K. Then τj-int(L)⊂K∩L. Also τj-int(L) ⊂ K∩L⊂L and L is (i, j)-ws-open. Seeing that K∩L is also (i, j)-ws-open.

**Theorem 7.2.37:** Each singleton point in space(P, τ1, τ2) is neither (i, j)-ws-open nor τi-w-open.

**Proof:** Take up(P, τ1, τ2) be a bitopological space. Assume x∈P. To prove {x} is either (i, j)-ws-open or τi-w-open. That is to prove P−{x}, is either (i, j)-ws-closed or τi-w-open, which follows from previous results.

**7.3 (τi, τj)-ws-open sets and their basic properties.**

In this section, we introduce (i, j)-ws-open sets in bitopological spaces and examine some of their properties.

**7.3.1 Definition:** Let i, j∈{1, 2} be fixed integers. In a bitopological space (P, τ1, τ2), a subset D ⊂ P is said to be (τi, τj)-ws-open (briefly, (i, j)-ws-open) if Dc is (i, j)-ws-closed.

**7.3.2 Theorem:** In a bitopological space (P, τ1, τ2), we have the following

1. Each (i, j)-g#-open set is (i, j)-ws-open but inverse is untrue.
2. Each (i, j)- g\*-open set is (i, j)-ws-open but inverse is untrue.
3. Each (i, j)--open set is (i, j)-ws-open but inverse is untrue.
4. Each (i, j)--open set is (i, j)-rw-open but inverse is untrue.
5. Each (i, j)-rb-open set is (i, j)-rw-open but inverse is untrue.
6. Each (i, j)-ws-open set is (i, j)-gspr-open but inverse is untrue.
7. Each (i, j)-ws-open set is (i, j)-gsp-open but inverse is untrue.
8. Each (i, j)-ws-open set is (i, j)-rgb-open but inverse is untrue.

**Proof:** Proof is obtained from the Theorems 7.2.3, to 7.2.18.

**7.3.3 Theorem:** If D and Eare (i, j)-ws-open sets, then DE is (i, j)- ws-open.

**Proof:** The result is obtained from theorem 7.2.25.

**7.3.4 Remark:** The intersection of two (i, j)-ws-open sets is generally not an (i, j)-ws-open set as seen from example 7.3.5.

**7.3.5 Example** Take P={ 1, 2, 3}, τ1={P, φ, {1}, {2,3}} and τ2={ P, φ, {1}, {2}, {1, 2}}. Then the subsets {1, 3} and {1, 3} are (1, 2)-ws-open sets, but {1, 3}{2,3}={3} is not (1, 2)-ws-open set in the bitopological space (P, τ1, τ2).

**7.3.6 Theorem:** A subset D of(P, τ1, τ2) is (i, j)-ws-open iff F⊂ τj-int(D), whenever F is τi-w-open and F⊂ D.

**Proof:** Pretend that F⊂ τi-int(D) whenever F⊂ D and F is τi-w-open. To show that D is (i, j)-ws-open. Assume that G be τi-w-open and Dc ⊂ G. Thun GC⊂ D and GC is τi-w-open, by Lemma 1.2.5. By hypothesis, GC⊂ τj-int(D). That is (τj-int(D))c⊂ G, since τj-scl(Dc)=(τj-int(D))c. Thus Dc is (i, j)-ws-closed. That is D is (i, j)-ws-open.

Inversly, Pretend that D is (i, j)-ws-open, F⊂D and F is τi-w-open. Then Dc ⊂ Fc and FC is also τi-w-open by Lemma 1.2.5. Seeing that Dc is (i, j)-ws-closed, we have τj-scl(Dc) ⊂ Fc and so F⊂ τj-int(D), since τj-scl(Dc) = (τj-int(D))c.

**7.3 (τi, τj)-ws-closure in bitopological spaces.**

W. Dunham [33] developed the concept of generalized closure operator C\* and Fukutake [34] derived and examined the concept of pairwise generalized closure operator (τi, τj)-cl\* in a bitopological spaces. Analogous to that we introduce the pairwise ws-closure operator “ (i, j)-ws-cl ” in bitopological spaces and examine some of their related properties.

**Definition 7.3.1:** Let (P, τ1­,τ2) be a bitopological space and i, j∈{1, 2} be fixed integers. For each subset E of P, define (τi, τj)-ws-cl(E)= ∩{G: E ⊂ G∈Dws(i, j)} (briefly: (i, j)-ws-cl(E)).

**Theorem 7.3.2:** If G and H be subsets of (P, τ1­,τ2). Then

(1) (i, j)-ws-cl(P)=P and (i, j)-ws-cl(φ)=φ.

(2) G ⊂ (i, j)-ws-cl(G).

(3) If H is any (i, j)-ws-closed set containing G, then (i, j)-ws-cl(G)⊂H.

**Proof:** Obtained form the Definition 7.3.1.

**Theorem 7.3.3:** Let G and H be subsets of (P, τ1­,τ2) and i, j∈{1, 2} be fixed integers. If G ⊂ H, then (i, j)-ws-cl(G) ⊂ (i, j)-ws-cl(H).

**Proof:** Take up G⊂H. By Definition 7.3.1, (i, j)-ws-cl(H)=∩{F:H⊂F∈Dws(i, j)} If H⊂F∈Dws(i, j), since G⊂H, G⊂H⊂F∈Dws(i, j), we have (i, j)-ws-cl(G) ⊂ F. Henceforth (i, j)-ws-cl(G) ⊂ ∩{F:H⊂F∈Dws(i, j)}= (i, j)-ws-cl(H). That is (i, j)-ws-scl(G) ⊂ (i, j)-ws-cl(H).

**Theorem 7.3.4:** Let G be a subset of (P, τ1­,τ2). If τ1­,⊂ τ2 and WO(P, τ1) ⊂ WO(P, τ2), then (1, 2)-ws-cl(G)⊂ (2, 1)-ws-cl(G).

**Proof:** By definition 7.3.1, (1,2)-ws-cl(G)=∩{F:G⊂ F∈Dws(1, 2)}. Seeing that τ1­,⊂ τ2, by Theorem 7.2.34, Dws(2, 1) ⊂ Dws(1, 2). Therefore ∩{F:G⊂ F∈Dws(1, 2)}⊂ ∩{F:G⊂ F∈Dws(2, 1)}. That is (1, 2)-ws-cl(G)⊂ ∩{F:G⊂ F∈Dws(2, 1)}= (2, 1)-ws-cl(G). Hence (1, 2)-ws-cl(G)⊂ (2, 1)-ws-cl(G).

**Theorem 7.3.5:** Let G be a subset of (P, τ1­,τ2) and i, j∈{1, 2} be fixed integers, then G ⊂ (i, j)-ws-cl(G) ⊂ τj-scl(G).

**Proof:** By Definition 7.3.1, it is obtained that G ⊂ (i, j)-ws-cl(G). Now to prove that (i, j)-ws-cl(G) ⊂ τj-scl(G). By definition of closure, τj-scl(G)=∩{F⊂P: G⊂F and F is τj-closed}. If G ⊂ F and F is τj-closed, then F is (i, j)-ws-closed, as each τj-closed set is is (i, j)-ws-closed. Henceforth (i, j)-ws-cl(G) ⊂ ∩{F⊂P: G⊂F and F is τj-closed}= τj-scl(G). Hence (i, j)-ws-cl(G) ⊂ τj-scl(G).

**7.4 Dws(i, j)-σ k-continuous maps and ws-bi-continuous maps.**

In this section a new class of maps called Dws(i, j)-σk-continuous maps in bitopological spaces are developed and examined. During this process, some of their related properties are obtained. Also, we form the concept ofws-bi-continuity and ws-s-bi-continuity in bitopological spaces and examine some of their related properties.

**Definition7.4.1:** A map h: (P, τ1, τ2)→(Q, σ1, σ2) is called Dws(i, j)-σ k-continuous if the inverse image of every σ k-closed set is an (i, j)-ws-closed set in (P, τ1, τ2).

**Remark7.4.2:** If τ1=τ2=τ and σ1=σ2=σ in Definition 7.4.1, then the Dws(i, j)-σk-continuity of maps coincides with ws-continuity of maps in topological spaces.

**Theorem7.4.3:** If a map h :(P, τ1, τ2)→(Q, σ1, σ2) is τj-σk-continuous, then it is a Dws(i, j)-σk-continuous.

**Proof:** Take L be a σk-closed set. Since h is τj-σ k-continuous, h−1(L) is τj-closed. By Theorem 7.2.7, h−1(L) is (i, j)-ws-closed in (P, τ1, τ2). Therefore h is Dws(i, j)-σ k-continuous.

We use example 7.4.4 to prove the inverse of theorem is untrue.

**Example 7.4.4:** Let P={1, 2, 3, 4}, τ1={P, φ, {1},{2},{1,2},{1,2,3} and τ2={P, φ, {1},{1, 2}{1, 2, 3}}, Q={1, 2, 3}, σ1={Q, φ, {1},{2},{1,2}} and σ2={Q, φ, {1},{2,3}}. Define a map h: (P, τ1, τ2)→(Q, σ1, σ2) by h(1)=4,h(2)=2 ,h(3)=2, ,h(4)=3. Then h is Drw(1, 2)-σ2-continuous but it is not τ1-σ2-continuous, Seeing that for the σ2-closed set {2,3}, h−1({2,3})={2, 3} is not τ1-closed.

**Theorem 7.4.5:** Map h: (P, τ1, τ2)→(Q, σ1, σ2) is (i, j)- σk -g#-continuous, then it is Dws(i, j)-σk-continuous.

**Proof:** Let L be a σ k-closed set. Since h is (i, j)-σk-g#-continuous, h−1(L) is (i, j)- g#-closed. By Theorem 7.2.3, h−1(L) is (i, j)-ws-closed in (P, τ1, τ2). Therefore h is Dws(i, j)-σk-continuous.

We use example 7.4.6 to prove the inverse of theorem is untrue.

**Example 7.4.6:** Let P={1, 2, 3}, τ1={P, φ, {1},{2,3}} and τ2={P, φ, {1},{2},{1, 2}} Q={1, 2, 3}, σ1={Q,φ,{1},{2},{1,2},{1,3}} and σ2={Q,φ,{1}}, (1,2)- Dws ={ P, φ, {1},{2},{3}{2,3},{1,3}}, and (1,2)- g# ={ P, φ, {3},{2,3},{1,2}}, Define a map h: (P, τ1, τ2)→(Q, σ1, σ2) by h(1)=1, h(2)=2, h(3)=3. Then h is Dws(1, 2)-σ1-continuous but it is not (i, j)-σk-g#-continuous. Seeing that for the σ1-closed set {2}, h−1({2})={2} is not (1, 2)- g#-closed set.

**Theorem 7.4.7:** Map h: (P, τ1, τ2)→(Q, σ1, σ2) is (i, j)-σk-g\*-continuous, then it is Dws(i, j)-σk-continuous.

**Proof:** Assume L be a σ k-closed set. Seeing that h is (i, j)-σk- g\*-continuous, h−1(L) is (i, j)- g\*-closed. By Theorem 7.2.7, h−1(L) is (i, j)-ws-closed in (P, τ1, τ2). Hnceforth h is Dws(i, j)-σk-continuous.

We use example 7.4.8 to prove the inverse of theorem is untrue.

**Example 7.4.8:** Let P={1,2,3}, τ1={P, φ, {1},{2,3} and τ2={P, φ, {1},{2},{1,2}} Q={1,2,3}, σ1={Q, φ,{1},{2},{1,2},{1,2}} and σ2={Q,φ,{2}}, (1,2)- Dws ={ P, φ, {1},{2},{3}{2,3},{1,3}}, and (1,2)- g\*={ P, φ, {1},{3},{2,3},{1,2}}, Define a map h: (P, τ1, τ2)→(Q, σ1, σ2) by h(1)=1, h(2)=2, h(3)=3. Then h is Dws(1, 2)-σ1-continuous but it is not (i, j)-σk- g\*-continuous. Seeing that for the σ1-closed set {2}, h−1({2})={2}is not (1, 2)- g\*-closed set.

**Theorem 7.4.9:** Map h :(P, τ1, τ2)→(Q, σ1, σ2) is (i, j)-σk--continuous, then it is Dws(i, j)-σk-continuous.

**Proof:** Take up L be a σ k-closed set. See h is (i, j)-σk- -continuous, h−1(L) is (i, j)- -closed. By Theorem 7.2.9, h−1(L) is (i, j)-ws-closed in (P, τ1, τ2). Hencefoth h is Dws(i, j)-σk-continuous.

We use example 7.4.10 to prove the inverse of theorem is untrue.

**Example 7.4.10:** Let P={1,2,3}, τ1={P, φ, {1},{2,3} and τ2={P, φ, {1},{2},{1,2}} Q={1,2,3}, σ1={Q,φ,{1},{2},{1,2},{1,3}} and σ2={Q,φ, {2}}, (1,2)- Dws ={ P, φ, {1},{2},{3}{2,3},{1,3}}, and (1,2)- ={ P, φ, {1},{2,3}}, Define a map h: (P, τ1, τ2)→(Q, σ1, σ2) by h(1)=1, h(2)=2, h(3)=3. Then h is Dws(1, 2)-σ1-continuous but it is not (i, j)-σk- –continuous. Seeing that for the σ1-closed set {2}, h−1({2})={2}is not (1, 2)- closed set.

**Theorem 7.4.11:** Map h:(P, τ1, τ2)→(Q, σ1, σ2) is (i, j)-σk--continuous, then it is Dws(i, j)-σk-continuous.

**Proof:** Take L be a σ k-closed set. Seeing that h is (i, j)-σk- -continuous, h−1(L) is (i, j)- -closed. By Theorem 7.2.11, h−1(L) is (i, j)-ws-closed in (P, τ1, τ2). Henceforth h is Dws(i, j)-σk-continuous.

We use example 7.4.12 to prove the inverse of theorem is untrue.

**Example 7.4.12:** Let P={1,2,3}, τ1={P, φ, {1},{2,3} and τ2={P, φ, {1},{2},{1,2}} Q={1,2,3}, σ1={Q,φ,{1},{2},{1,2},{1,2}} and σ2={Q,φ, {2}}, (1,2)- Dws ={ P, φ, {1},{2},{3}{2,3},{1,3}}, and (1,2)- ={ P, φ, {1},{2,3}}, Define a map h: (P, τ1, τ2)→(Q, σ1, σ2) by h(1)=1, h(2)=2, h(3)=3. Then h is Dws(1, 2)-σ1-continuous but it is not (i, j)-σk- –continuous. Seeing that for the σ1-closed set {2}, h−1({2})={2} which is not (1, 2)- closed set.

**Theorem 7.4.13:** Map h:(P, τ1, τ2)→(Q, σ1, σ2) is (i, j)-σk-rb-continuous, then it is Dws(i, j)-σk-continuous.

**Proof:** Let L be a σ k-closed set. Since h is (i, j)-σk- rb -continuous, h−1(L) is (i, j)- rb-closed. By Theorem 7.2.13, h−1(L) is (i, j)-ws-closed in (P, τ1, τ2). Henceforth h is Dws(i, j)-σk-continuous.

We use example 7.4.14 to prove the inverse of theorem is untrue.

**Example 7.4.14:** Let P={1,2,3}, τ1={P, φ, {1},{2,3} and τ2={P, φ, {1},{2},{1,2}} Q={1,2,3}, σ1={Q, φ,{1},{2},{1,2},{1,2}} and σ2={Q,φ,{2}}, (1,2)- Dws ={ P, φ, {1},{2},{3}{2,3},{1,2}}, and (1,2)- rb={ P, φ, {1},{2,3}}, Define a map h: (P, τ1, τ2)→(Q, σ1, σ2) by h(1)=1, h(2)=2, h(3)=3. Then h is Dws(1, 2)-σ1-continuous but it is not (i, j)-σk- rb –continuous. Seeing that for the σ1-closed set {2}, h−1({2})={2} which is not (1, 2)- rb closed set.

**Theorem 7.4.15:** Map h:(P, τ1, τ2)→(Q, σ1, σ2) is (i, j)-σk-\*g-continuous, then it is Dws(i, j)-σk-continuous.

**Proof:** Let L be a σ k-closed set. Since h is (i, j)-σk- \*g -continuous, h−1(L) is (i, j)- rb-closed. By Theorem 7.2.5, h−1(L) is (i, j)-ws-closed in (P, τ1, τ2). Henceforth h is Dws(i, j)-σk-continuous.

We use example 7.4.16 to prove the inverse of theorem is untrue.

**Example 7.4.16:** Let P={1,2,3}, τ1={P, φ, {1},{2,3} and τ2={P, φ, {1},{2},{1,2}} Q={1,2,3}, σ1={Q,φ,{1},{2},{1,2},{1,2}} and σ2={Q,φ, {2}}, (1,2)- Dws ={ P, φ, {1},{2},{3}{2,3},{1,3}}, and (1,2)- \*g ={ P, φ, {1},{2,3}}, Define a map h: (P, τ1, τ2)→(Q, σ1, σ2) by h(1)=1, h(2)=2, h(3)=3. Then h is Dws(1, 2)-σ1-continuous but it is not (i, j)-σk- \*g -continuous, Seeing that for the σ1-closed set {2}, h−1({2})={2} is not (1, 2)- \*g -closed set.

**Theorem 7.4.17:** Map h:(P, τ1, τ2)→(Q, σ1, σ2) is Dws(i, j)-σk-continuous, then it is (i, j)-σ k- gspr continuous.

**Proof:** Take L be a σk-closed set. Since h is Dws(i, j)-σ k-continuous, h−1(L) is (i, j)-ws-closed. By Theorem 7.2.15, h−1(L) is (i, j)-gspr-closed in (P, τ1, τ2). Henceforth h is (i, j)-σk-gspr continuous.

We use example 7.4.18 to prove the inverse of theorem is untrue.

**Example 7.4.18:** Let P={1,2,3}, τ1={P, φ, {1}, {2,3}} τ2={P, φ, {1},{2},{1,2}} and Q={1,2,3}, σ1={Q, φ, {3}} and σ2={Q, φ}. (1, 2)- Dws ={ P, φ, {1},{2},{3}{2,3},{1,3}}, and (1,2)- gspr ={ P(P)}, Define a map h:(P, τ1, τ2)→(Q, σ1, σ2) by h(1)=1, h(2)=2 and h(3)=3, Then h is (1, 2)-σ1-gspr-continuous but it is not Dws(1, 2)-σ1-continuous. Seeing that for the σ1-closed set {1,2}, h−1({1,2})={1,2} is not a (1, 2)-ws-closed set.

**Theorem7.4.19:** Map h:(P, τ1, τ2)→(Q, σ1, σ2) is Dws(i, j)-σk-continuous, then it is (i, j)-σ k- gsp continuous.

**Proof:** Let L be a σk-closed set. Since h is Dws(i, j)-σ k-continuous, h−1(L) is (i, j)-ws-closed. By Theorem 7.2.17, h−1(L) is (i, j)-gsp-closed in (P, τ1, τ2). Henceforth h is (i, j)-σk-gsp continuous.

We use example 7.4.20 to prove the inverse of theorem is untrue.

**Example 7.4.20:** Let P={1,2,3}, τ1={P, φ, {1}, {2,3}} τ2={P, φ, {1},{2},{1,2}} and Q={1,2,3}, σ1={Q, φ, {3}} and σ2={Q, φ}.(1,2)- Dws ={ P, φ, {1},{2},{3}{2,3},{1,3}}, and (1,2)- gsp ={ P(P)}, Define a map h:(P, τ1, τ2)→(Q, σ1, σ2) by h(1)=1, h(2)=2 and h(3)=3, Then h is (1, 2)-σ1-gsp-continuous but it is not Dws(1, 2)-σ1-continuous. for the σ1-closed set {1,2}, h−1({1,2})={1,2} is not a (1, 2)-ws-closed set.

**Theorem 7.4.21:** Map h:(P, τ1, τ2)→(Q, σ1, σ2) is Dws(i, j)-σk-continuous, then it is (i, j)-σ k- rgb continuous.

**Proof:** Let L be a σk-closed set. Since h is Dws(i, j)-σ k-continuous, h−1(L) is (i, j)-ws-closed. By Theorem 7.2.19, h−1(L) is (i, j)-rgb-closed in (P, τ1, τ2). Henceforth h is (i, j)-σk-rgb continuous.

We use example 7.4.22 to prove the inverse of theorem is untrue.

**Example 7.4.22:** Let P={1,2,3}, τ1={P, φ, {1}, {2,3}} τ2={P, φ, {1},{2},{1,2}} and Q={1,2,3}, σ1={Q, φ, {3}} and σ2={Q, φ}. (1,2)- Dws ={ P, φ, {1},{2},{3}{2,3},{1,2}}, and (1,2)- rgb ={ P(P)},Define a map h:(P, τ1, τ2)→(Q, σ1, σ2) by h(1)=1, h(2)=2 and h(3)=3, Then h is (1, 2)-σ1-rgb-continuous but it is not Dws(1, 2)-σ1-continuous. Seeing that for the σ1-closed set {1,2}, h−1({1,2})={1,2} which is not a (1, 2)-ws-closed set.

**Remark 7.4.23:** Dws(i, j)-σ k-continuous maps and Drw(i, j)-σ k-continuous((i, j)-σ k-R\*continuous) maps are independent.

**Example 7.4.24:** Let P={1,2,3}, τ1={P, φ, {1}, {2,3}} and τ2={P, φ, {1},{2},{1,2}} and Q={1,2,3}, σ1={Q, φ,{1},{2},{1,2},{1,2}} and σ2={Q, φ}. Define a map h:(P, τ1, τ2)→(Q, σ1, σ2) by h(1)=1,h(2)=2 and h(3)=3. Then this function h is Dws(1, 2)-σ2-continuous but it is not Drw(1, 2)-σ1-continuous((i, j)-σk-R\*continuous), Seeing that for the σ1-closed set {2}, h−1({2})={2} is not (1, 2)-rw-closed(R\*-closed ) in (P, τ1, τ2).

**Example 7.4.25:** Let P={1,2,3}, τ1={P, φ, {1}, {2}, {1,2}} and τ2={P, φ, {1}, {2},{1,2},{1,3}} and Q={1,2,3}, σ1={Q, φ, {3}}. and σ2={Q, φ, }. Define a map h:(P, τ1, τ2)→(Q, σ1, σ2) by h(1)=1, h(2)=2 and h(3)=3. Then this function h is Drw (1, 2)-σ1-continuous((i, j)-σ1-R\*continuous) but it is not Dws(1, 2)-σ1-continuous. Seeing that for the σ-closed set {1,2}, h−1({1,2})={1,2} is not (1, 2)-ws-closed in (P, τ1, τ2).

**Remark 7.4.26:** From the above discussions and known results we have the following implications.

(i, j)-σ k -gsp continuous

(i, j)-σ k-g\*- continuous

(i, j)-σ k -rgb- continuous spaces

Drw(i, j)-σ k-continuous

(i, j)-σ k-- continuous

(i, j)-σ k -g#- continuous

(i, j)-σ k -gspr- continuous

Dws(i, j)-σ k-continuous

(i, j)-σ k-gp- continuous

(i, j)-σ k -\*g- continuous

(i, j)-σ k -R\*- continuous

Note: A B Means A implies B, but inverse is untrue.

A B Means A and B are independent.

**Theorem 7.4.28:** The h continuous llowing statements are one and the same.

(1) A map h:(P, τ1, τ2)→(Q, σ1, σ2) is Dws(i, j)-σ k-continuous.

(2) The inverse image of every σ k-open set in Q is (i, j)-ws-open in P.

**Proof:** (1) ⇒ (2) Assume L be a σ k-open set in Q. Then L C is σ k-closed set in Q. Seeing that h is Dws(i, j)-σ k-continuous, h−1(L C) is (i, j)-ws-closed in P. That is h−1(L C)=(h−1(L ))C and so h−1(L ) is (i, j)-ws-open in (P, τ1, τ2).

(2)⇒ (1) Let M be a σ k-closed set in Q. Then LC is σ k-open set in Q. By hypothesis, h−1(MC) is (i, j)-ws-open in P. That is h−1(MC) = (h−1(M))C and so h−1(M) is (i, j)-ws-closed in (P, τ1, τ2). Henceforth h is Dws(i, j)-σk-continuous.

**Theorem 7.4.29:** Map h:(P, τ1, τ2)→(Q, σ1, σ2) is Dws(i, j)-σk-continuous and g: (Q, σ1, σ2)→(R, η1, η2) is σk-ηn-continuous, then gοh is Dws(i, j)-ηn-continuous.

**Proof:** Let K be ηn-closed set in (R, η1, η2). Seeing that g is σk-ηn-continuous, g−1(K) is a σk-closed set in (Q, σ1, σ2). Seeing that h is Drw(i, j)-σk-continuous, h−1(g−1(K))= (gοh)−1(K) is a (i, j)-ws-closed set in (P, τ1, τ2) and hence gοh is Dws(i, j)-ηn-continuous.

**Definition 7.4.30:** (i) A map h:(P, τ1, τ2)→(Q, σ1, σ2) is called ws-bi- continuous if h is Dws(1, 2)-σ2-continuous and is Dws(2, 1)-σ1-continuous..

(ii) A map h:(P, τ1, τ2)→(Q, σ1, σ2) is called rw-strongly-bi-continuous (briefly rw-s-bi-continuous) if h is rw-bi-continuous, Drw(2, 1)-σ2-continuous and Drw(1, 2)-σ1-continuous.

**Theorem7.4.31:** Let h: (P, τ1, τ2)→(Q, σ1, σ2) be a map.

(1) If h is bi-continuous then h is ws-bi-continuous.

(2) If h is s-bi-continuous then h is rw-s-bi-continuous.

**Proof:** (i) Take h:(P, τ1, τ2)→(Q, σ1, σ2) be a bi-continuous map. Then h is τ1-σ1-continuous and τ2-σ2-continuous and so by Theorem 7.4.3, h is Dws(1, 2)-σ2-continuous and Dws(2, 1)-σ1-continuous. Thus h is ws-bi-continuous.

(ii) Similar to (i), using Theorem 7.4.3.

We use example 7.4.32 to prove the inverse of theorem is untrue.

**Example 7.4.32:** Let P={1,2,3}, τ1={P, φ, {1},{2,3}} and τ2={P, φ,{1},{2} {1,2}} and Q={1, 2,3}, σ1={Q, φ, {1}} and σ2={Q, φ, {1},{2},{1,2},{1,3}}. Define a map h: (P, τ1, τ2)→(Q, σ1, σ2) by h(1)=1 ,h(2)=2,h(3)=3. Then h is ws-bi-continuous but not bi-continuous.

**Example 7.4.33:** Let P={1,2,3}, τ1={P, φ, {1},{2,3}} and τ2={P, φ,{1},{2} {1,2}} and Q={1,2,3}, σ1={Q, φ, {2}} and σ2={Q, φ, {1}}. Define a map h: (P, τ1, τ2)→(Q, σ1, σ2) by h(1)=1 ,h(2)=2, h(3)=3. Then h is ws-s-bi-continuous but not s-bi-continuous.

**CHAPTER-8**

**On Weakly semi closed separation axioms in topological spaces**

**8.1. Introduction:**

From the literature survey on separation axioms, we observed that there is an signiﬁcant work on weak forms of separation axioms like Tk spaces (k=0, 1, 2), normal and regular axioms in particular, several other neighbouring forms of them have been elaborated in many articals. In the year 1975, Maheshwari and Prasad [6] introduced a new class of spaces called s-normal space using semi open sets. Using the concept of generalized closed sets of Levine [5], T. Noiri and Papa [7] introduced g-regular and g-normal spaces in topological spaces. Sheik John [8] introduced and studied w-regular and w-normal spaces. Recently, Basavaraj M. Ittanagi et al [1, 2, and 3] introduced and studied basic properties of ws-closed sets, ws-continuous functions and ws-closed maps in topological spaces. In this paper, using the notion of ws-open sets, ws-separation axioms are introduced and studied. Also the relationships with some other functions are discussed.

This chapter has four sections. In second section of this chapter, a new class of space known as ws- separation axioms namely ws-T0 space, ws-T1 space, ws-T2 space, are introduced and studied. It is observed that each α- T0 is ws-T0 space, each α- T1 is ws-T1 space and each α- T2 is ws-T2 space, few of its major results are obtained.

In third section, we innovate the concepts of ws-regular spaces. It is observed that each ws-regular spaces are regular. Characterizations of ws-regular spaces in topological space are obtained.

In fourth section, we establish the concept of ws-normal spaces by deliberating its characterization with its few properties.

* 1. **ws-Tk Space (k=0,1,2)**

**Definition 8.2.1:** A topological space P is called an ws-T0 if for each pair of distinct points 1, 2 of P, ∃an ws-open sets H in P containing one of them and not the other.

**Example 8.2.2:** Let P= {1, 2, 3, 4} and τ ={P, ϕ, {1},{2},{1,2},{1,2,3}}Then (P, τ) is ws-T0 space. Seeing that for two distinct points ‘1’ and ‘2’ of (P, τ) there exist an ws-open set {1} ∋: 1 ∈ {1}, 2 ∉ {1}

**Theorem 8.2.3:**

* 1. T0-space is ws-T0 space
  2. semi -T0space is ws-T0 space
  3. α-T0 space is ws-T0 space
  4. g#-T0space is ws-T0 space
  5. -T0space is ws-T0 space
  6. g -T0space is ws-T0 space
  7. g#s -T0space is ws-T0 space
  8. rb -T0space is ws-T0 space
  9. regular-T0space is ws-T0 space

**Proof:**

1. Let P is a T0 space. For each pair of disjoint points a, b of P open set H in P a ∈ H, b ∉ H and b ∈ H, a ∉ H. But every open set is ws-open then ∃ws-open set H in P a ∈ H, b ∉ H and b∈ H, a ∉ H. Henceforth P is a ws-T0 space.

Similarly the remaining (ii to ix) results can be proved

We use example 8.2.4 to prove the inverse of theorem is untrue.

**Example 8.2.4:** Let P= {1,2,3,4} and τ ={P, ϕ, {1}, {1,2},{1, 2 ,3}}.Then g#-(respectively -, \*g )open (P, τ)={ P, ϕ, {1}, {1,2},{1, 2,3}}.-(semi-open,, g#s, \*g)open (P, τ) ={P, ϕ, {1}, {1,2},{1,3},{1,4},{1,2,3},{1,2,4},{1,3,4}}, rb-open={ P, ϕ, {1}}, ws -open (P, τ)={P, ϕ, {1},{2},{3},{1,2},{1,3},{1,4},{2,3},{1, 2, 3},{1,2, 3},{1, 3, 4}}. Here (P, τ) is ws-T0 space but it is not T0-space, g# -T0 space, -T0 space, \*g -T0 space , -T0 space , g#s -T0 space, g -T0 space and rb-T0 space. Seeing that for two distinct points ‘1’ and ‘2’ there do not exist open set, α-open set, regular-open set G in P containing one of them and not the other.

**Theorem 8.2.5:** Let P is a topological space and Q is an ws-T0 space. If h: P → Q is injective and ws– irresolute then P is ws-T0 space.

**Proof:** Suppose a, b ∈ P a ≠ b. Seeing that h is injective thus f(a) ≠ f(b). Also Q is

ws-T0 space then a ws-open sets M in Q f(a) ∈ M , f(b) ∉ M or a ws-open sets N in Q f(b) ∈ N, f(a) ∉ N with f(a) ≠ f(b). Seeing that h is

ws- Irresolute thus h–1(M) is a ws-open sets in P a ∈ h–1(M), b ∉ h–1(M) or h–1(N) is a ws-open sets in P b ∈ h–1(N), a ∉ h–1(N). Hence P is ws-T0 space.

**Theorem 8.2.6**: (P, τ) is ws-T0 space iff for each pair of distinct a, b of P,

ws-cl({1}) ≠ ws-cl({2}) .

**Proof:** Take (P, τ) be ws-T0 space. Assume a, b ∈ P a ≠ b, thus a ws-open set N containing one of the points but not the other, say a ∈ N and b ∉ N. Then NC is ws-closed containing b but not a. But ws-cl({2}) is the smallest ws-closed set containing b. Henceforth ws-cl({2}) ⊂Nc and hence a∉ ws-cl({2}). Thus ws-cl({1}) ≠ ws-cl({2}).Inversely , Pretend a, b ∈ P, a≠ b and ws-cl({1}) ≠ ws-cl({2}). Assume c∈ P c∈ ws-cl({1}) but c ∉ ws-cl({2}). If a ∈ ws-cl({2}) then ws-cl({1}) ⊂ ws-cl({2}) and hence c ∈ ws-cl({2}). This is a contradiction. Henceforth x∉ ws-cl({2}). That is a ∈ (ws-cl(b)) c. Henceforth (ws-cl({2}))c is ws-open set containing a but not b. Hence (P, τ) is ws-T0 space.

**Definition 8.2.7:** A topological space P is called a ws-T1 if for each pair of distinct points a, b of P, a ws-open sets H1, H2 in P ∋: a ∈ H1, b ∉ H1 and b ∈ H2, a ∉ H2.

**Example 8.2.8:** Let P= {1, 2, 3} and τ ={P, ϕ, {1}, {2, 3}}. Then (P, τ) is ws-T1 space. Seeing that for two distinct points ‘1’ and ‘2’ of (P, τ) an ws-open set {1}, {2} ∋: 1 ∈ {1}, 2 ∉ {1} and 2 ∈ {2}, 1 ∉ {2}.

**Theorem 8.2.9:**

1. T1-space is ws- T1 space
2. semi - T1 space is ws- T1 space
3. α- T1 space is ws- T1 space
4. g#- T1 space is ws- T1space
5. - T1 space is ws- T1 space
6. g -T1 space is ws-T1 space
7. g#s -T1 space is ws-T1 space
8. rb –T1 space is ws-T1 space
9. regular-T1 space is ws-T1 space

**Proof:**

1. Take P is a T1 space. For each pair of disjoint points a, b of P open set H1, H2 in P a ∈ H1, b ∉ H1 and a ∈ H2, b ∉ H2. But each open set is ws-open then ws-open set H1, H2 in P a ∈ H1, b ∉ H1 and b ∈ H2, a ∉ H2 Henceforth P is a ws-T1 space.

Similarly the remaining (ii to ix) results can be proved

We use example 8.2.10 to prove the inverse of theorem is untrue.

**Example 8.2.10:** Let P= {1,2,3,4} and τ ={P, ϕ, {1}, {1, 2},{1, 2, 3}}.Then g#-(respectively -, \*g )open (P, τ)={ P, ϕ, {1}, {1,2},{1, 2 ,3}}.-(semi-open,, g#s, \*g)open (P, τ) ={P, ϕ, {1}, {1, 2},{1, 3},{1, 4},{1, 2, 3},{1, 2, 4},{1, 2, 4}}, rb-open={ P, ϕ, {1}}, ws -open (P, τ)={P, ϕ, {1},{2},{3},{1, 2},{1, 3},{1, 4},{2 ,3},{1, 2, 3},{1, 2, 4},{1, 2, 4}}. Here (P, τ) is ws-T1 space but it is not T1-space, g# -T1 space, –T1 space , \*g –T1 space , -T1 space , g#s –T1 space, g -T1 space and rb-T1 space. Seeing that for two distinct points ‘1’ and ‘2’ there do not exist open set, g# - open, – open, \*g – open, - open, g#s – open, g - open and rb- open set H1, H2 in P ∋: 1 ∈ H1, 2 ∉ H1 and 2 ∈ H2, 1 ∉ H2.

**Theorem 8.2.11:** Let P is a topological space and Q is a ws-T1 space. If h: P → Q is injective and ws-irresolute then P is ws-T1 space.

**Proof:** Theorem 8.2.5

**Theorem 8.2.12:** If (P, τ) is ws-T1 space then (P, τ) is ws-T0 space.

**Proof:** Take (P, τ) be a ws-T1 space. For each pair of disjoint points a, b of (P, τ) open sets H1, H2 in (P, τ) a ∈ H1, b ∉ H1 and b ∈ H2, a ∉ H2. Hence we have

a ∈ H1, b ∉ H1. Henceforth (P, τ) is a ws-T0 space.

**Theorem 8.2.13**: A topological space P is ws-T1 space iff x∈ P singleton {1} is ws-closed set in P.

**Proof:** Take P be ws-T1 space and Assume a ∈ P, to show that {1} is ws-closed set. We will show P– {1} is ws-open set in P. take b ∈ P–{1}, implies a ≠ b ∈ P and seeing that P is ws-

T1 space thus two ws-open sets H1, H2 a ∉ H1, b ∈ H2 ⊆ P–{1}.

Seeing that b∈ H2 ⊆ P–{1} then P–{1} is ws-open set. Hence {1} is ws-closed set.

Inversely, let a ≠ b ∈ P then {1}, {2} are ws-closed sets. That is P–{1} is ws-open set. Clearly, a ∉ P–{1} and b ∈ P–{2}.Similarly P–{2} is ws-open set, b ∉ P–{2} and

a ∈ P–{2}. Hence P is ws-T1 space.

**Definition 8.2. 14:** A topological space P is called a ws-T2 if for each pair of distinct points a, b of P, a ws-open sets H1, H2 in P ∋: a ∈ H1, b ∈ H2 and H1∩H2= ϕ.

**Example 8.2.15:** Let P= {1,2,3} and τ ={P, ϕ, {1},{2}, {1,2}}. Then (P, τ) is ws-T2 space. Seeing that for a pair of distinct points ‘1’ and ‘2’ of (P, τ) there exist an ws-open set {1}, {2} ∋: 1 ∈ {1}, 2 ∈ {2} and {1} ∩{2} = ϕ.

**Theorem 8.2.16:**

1. Each T2-space is ws- T2 space
2. Each semi – T2 space is ws- T2 space
3. Each α- T2 space is ws- T2 space
4. Each g#- T2 space is ws- T2 space
5. Each - T2 space is ws- T2 space
6. Each g -T2 space is ws-T2 space
7. Each g#s -T2 space is ws-T2 space
8. Each rb –T2 space is ws-T2 space
9. Each regular-T2 space is ws-T2 space

**Proof:**

1. Let P is a T2 space. pair of disjoint points a, b of P open set H1, H2 in P a ∈ H1, b ∈ H2 and H1∩H2= ϕ. But every open set is ws-open then ws-open set H1, H2 in P a ∈ H1, b ∈ H2 and H1∩H2= ϕ. Henceforth P is a ws-T2 space.

Similarly the remaining (ii to ix) results can be proved

**Theorem 8.2.17:** If (P, τ) is ws-T2 space then (P, τ) is ws-T1 space.

**Proof:** Take up (P, τ) be a ws-T2 space. pair of disjoint points a, b of (P, τ) disjoint open sets K and L in (P, τ) a ∈ K and b ∈ L. Hence we have a ∈ K, b ∉ K and a ∈ L, b ∉ L . Henceforth (P, τ) is a ws-T1 space.

**Theorem 8.2.18:** The given Statements are one and the same., for space (P, τ),

* 1. (P, τ) is ws-T2 space.
  2. If a ∈ P, then a≠ b, there will be ws-open set K containing a b ∉ ws-cl (K).

**Proof**:

(i) ⟹ (ii) Let a ∈ P. If b ∈ P is b ≠ a, disjoint ws-open sets K and L a∈ K and b∈ L. Then a ∈ K ⊂P–L which implies P–L is ws-open and

b ∉ P–L. Henceforth b ∉ ws-cl (K).

(ii) ⟹ (i) Take up a, b ∈ P and a ≠b. By (ii), a ws-open K containing a

b ∉ ws-cl(K). Henceforth b∈ P–(ws-cl(K)). P–(ws-cl(K)) is ws-open and a ∈ P– (ws-cl(K)). Also K∩P–(ws-cl(K)) = Φ. Hence (P, τ) is ws-T2 space.

**Theorem 8.2.19**: Pretend P be a topological space and Q is ws-T2 space. If h: P → Q is injective and ws-irresolute then P is ws-T2 space.

**Proof:** Pretend a, b ∈ P a ≠ b. Seeing that h is injective, implies h(a) ≠ h(b). Also Q is

ws- T2 space thus there are two ws-open sets M and N in Q h(a) ∈ M, h(b) ∈ N and M∩N=Φ. Seeing that h is ws-irresolute thus h–1(M), h–1(N) are two ws-open sets in P,

a ∈ h–1(M), b ∈ h–1(N), h–1(M) ∩ h–1(N) = Φ. Hence P is ws-T2 space.

**Theorem 8.2.20:** Pretend P is a topological space and Q is a T2 space. If h: P → Q is injective and ws-continuous then P is ws-T2 space.

**Proof:** Pretend a, b ∈ P a ≠ b. Seeing that h is injective, implies f(a) ≠ f(b). Also Q is an T2 space, thus there are two open set M and N in Q, h(a) ∈ M, h(b) ∈ N and

M∩N =Φ. seeing that h is ws-continuous thus h–1(M), h–1(N) are two ws-open sets in P.

Then a ∈ h–1(M), b ∈ h–1(M), h–1(M) ∩ h–1(N) =Φ. Hence P is ws-T2 space.

**8.3 ws–Regular Space**

In this section, we determine a new kind of spaces called ws-regular spaces using ws-closed sets and verified some of their related characterizations.

**Definition 8.3.1**: A topological space P is termed as ws-regular if for each ws-closed set F and a point a ∉ F, disjoint open sets G and H ∋: F ⊆ G and P ∈ H. We have the following interrelationship between ws-regularity and regularity.

**Theorem 8.3.2:** Each ws-regular space is regular.

**Proof:** Take up P be ws-regular space. Assume F be any closed set in P and a point a∈ P a ∉ F. and F is ws-closed and a ∉ F. Seeing that P is ws-regular space, a pair of disjoint open sets G and H ∋: F ⊆ G and a H. Hence P is a regular space.

**Theorem 8.3.3**: Pretend P is a regular space and Tws–space, then P is ws-regular.

**Proof:** Take up P be a regular space and Tws– space. Assume F be any ws-closed set in P and a point a ∈ P a∉ F. Seeing that P is Tws– space. F is closed and a ∉ F. Seeing that P is a regular space, ∃a pair of disjoint open sets G and H ∋: F ⊆ G and P H. Hence P is ws-regular space.

**Theorem 8.3.4:** Each subspace of a ws-regular space is ws-regular.

**Proof:** Take P be ws-regular space. Assume Q be a subspace of P. Let a ∈ Q and F be a ws-closed set in Q a ∈ F. Thus there is a closed set and so ws-closed set D of P with F = Q ∩ D and P D. Therefore we have a P, D is ws-closed in P a ∈ D. Seeing that P is ws-regular, open sets G and H a ∈ G, D⊆ H and G ∩ H = Φ. Note that Q ∩ G and Q ∩ H are open sets in Q. Also a ∈ G and a ∈ Q, which implies P ∈ Q ∩ G and D ⊆ H implies Q ∩ D ⊆ Q ∩ H, F ⊆ Q ∩ H. Also (Q ∩ G) ∩ (Q ∩ H) = Φ. Hence Q is ws-regular space. We have yet another characterization of ws-regularity in the following.

**Theorem 8.3.5:** Let P is a space. Pretend P is a ws-regular and a T1 space then

P is ws-T2 space.

**Proof:** Pretend a, b ∈ P a = b. Seeing that P is T1– space thus an open set M a ∈ M, b ∈ M. Seeing that P is ws-regular space and M is an open set which contains a, an ws-open set N ∋: a∈ N⊂ws-cl (N) ⊆ M. Seeing that b ∈ M,

hence b ∈ ws-cl(N). Therefore b ∈ P–(ws-cl (N)). Hence there are ws-open sets N and P–(ws-cl(N)) (P– (ws-cl(N)))∩ N = Φ. Hence P is ws-T2 space.

**Theorem 8.3.6**: Let h: P → Q be a bijective, ws-closed map from a topological space

P into a ws-regular space Q. If P is Tws–space, then P is ws-regular.

**Proof:** take up a ∈ P and F be ws-closed set in P with a ∈ F. Seeing that P is Tws–space, F is closed in P. Then h(F) is ws-closed set with h(P) ∈ h(F) in Q , Seeing that h is ws-closed. As Q is ws-regular, disjoint open sets M and N h(a) ∈ M and h(F) ⊆ N .Henceforth a ∈ h–1(M) and F ⊆ h–1 (N ). Hence P is ws-regular space.

**Theorem 8.3.7**: The given below statements are one and the same. for space P.

* + 1. P is ws-regular space.
    2. For each a ∈ P and each ws-open neighbourhood M of P ∃an open neighbourhood N of P ∋: cl (N) ⊆ M.

**Proof:**

(i) ⟹ (ii): Pretend P is a ws-regular space. Let M be any ws-neighbourhood of P. Then ∃ws-open set H a ∈ H ⊆ M. Now P– H is ws-closed set and a ∈P – H. Seeing that P is ws-regularopen sets M and N P – H ⊆ M, a∈ N and M∩N = ϕ and so N ⊆ P –M. Now cl(N) ⊆ cl(P – M) = P –M and P – H ⊆ M. This implies P – M ⊆ H ⊆ M. Therefore cl(N) ⊆ M.

(ii) ⟹ (i): Pretend F be any ws-closed set in P and a ∉ F or a ∈ P –F and P –F is a ws-open and so P –F is a ws-neighbourhood of P. By hypothesis, ∃an open neighbourhood N of a a ∈ N and cl(N) ⊆ P –F. This implies F ⊆ P –cl(N) is an open set containing F and N ∩ {(P –cl(N)} = ϕ. Hence P is ws-regular space. We have another characterization of ws-regularity in the following.

**Theorem 8.3.8**: P is topological space is ws-regular iff for each ws-closed set F of P and each a ∈ P –F open sets G and H of P a ∈ G, F ⊆ H and cl(G) ∩ cl(H) = ϕ.

**Proof:** Pretend P is ws-regular space. Assume F be a ws-closed set in P with a ∈ F. Then open sets M and H of P a ∈ M, F ⊆ H and M ∩H = ϕ. This implies

M ∩ cl (H) = ϕ. As P is ws-regular, open sets M and N a ∈ M, cl (H) ⊆ N and M ∩ N = Φ, so cl (M) ∩ N = Φ. Assume G = M ∩ M, then G and H are open sets of P ∋: a ∈ G, F ⊆ H and cl (H) ∩ cl (H) = ϕ. Inversely, if for each ws-closed set F of P and each a ∈ P– F open sets G and H a ∈ G, F ⊆ H and cl(H) ∩ cl(H) = Φ. This implies a ∈ G, F ⊆ H and G ∩ H = ϕ. Hence P is ws-regular. Now we prove that ws-regularity is a hereditary property.

**Theorem 8.3.9:** P is a topological space then results given below are one and the same.

1. P is ws-regular
2. For each a ∈ P and each ws-open set M in P ∋: P ∈ M ∃an open set V in P ∋: P ∈ N ⊆ cl(N ) ⊆ M
3. For each point a∈ P and for each ws-closed set A with P ∈ D, ∃an open set N containing P ∋: cl (N) ∩ D = Φ.

**Proof:**

(i) ⟹ (ii): Follows from Theorem 8. 3.5.

(ii) ⟹ (iii): Suppose (ii) holds. Assume a ∈ P and D be a ws-closed set of P ∋: a ∈ D. Then P –D is a ws-open set with a ∈ P –D. By hypothesis, an open set N ∋: P ∈ N ⊆ cl (N) ⊆ P –D. That is P ∈N, N ⊆ cl (D) and cl (D) ⊆ P –D. So P ∈ N and cl (N) ∩ D = Φ.

(iii)⟹ (ii): Let a ∈P and M is a ws-open set in P ∋: a ∈ M. Then P –M is a ws-closed set and a ∈P –M. Then by hypothesis, ∃ an open set N containing P ∋: cl (N) ∩ (P –M) = Φ. Therefore a ∈ N, cl (N) ⊆ M so P ∈ N ⊆ cl (N) ⊆ M. The invariance of ws-regularity is given in the following.

**Theorem 8.3.10:** Pretend h: P → Q be a bijective, ws-irresolute and open map from a ws-regular space P ino a topological space Q, then Q is ws-regular.

**Proof:** Pretend b ∈Q and F be a ws-closed set in Q with b ∈ F. Seeing that h is ws-irresolute,

h-1 (F) is ws-closed set in P. Let h(a) = b so that a = h–1(b) and a ∈ h–1(F). Again P is ws- regular space, open sets M and N ∋: a ∈ M and h–1 (F) ⊆ G, M ∩N = Φ. Seeing that h is open and bijective, we have b ∈ h(M) ,F ⊆ h(N ) and

h(M) ∩ h(N )= h(M ∩ N ) = h(Φ) = Φ. Hence Q is ws-regular space.

**8.4 ws-Normal Spaces**

In this section, we introduce the concept of ws-normal spaces and study some of their characterizations.

**Definition 8.4.1**: A topological space P is said to be ws-normal if pair of disjoint ws-closed sets D and E in P, ∃ a pair of disjoint open sets M and N in P ∋: D ⊆ M and E ⊆ N .We have the following interrelationship.

**Theorem 8.4 .2:** Each ws-normal space is normal.

**Proof:** Take P be a ws-normal space. Assume D and E be a pair of disjoint closed sets in P.

And D and E are ws-closed sets in P. Seeing that P is ws-normal, ∃ a pair of disjoint open sets G and H in P ∋: D ⊆ G and E ⊆ H. Hence P is normal.

We use the following example to prove the inverse of the theorem is untrue.

**Example 8.4.4:**Let Let P={1,2,3,4}, τ= {P, , {1},{2},{1,2},{1,2,3}} Then the space P is normal but not ws-normal, seeing that the pair of disjoint ws-closed sets namely,

D ={1} for which there do not exists disjoint open sets G and H D ⊆ G and E ⊆ H.

**Theorem 8.4.5:** A ws-closed subspace of a ws-normal space is ws-normal.

**Proof:** Take up P be ws-normal space. Assume Q be a ws-closed subspace of P. Pretend D and E be pair of disjoint ws-closed sets in Q. Thus D and E be pair of disjoint ws-closed sets in P. seeing that P is ws-normal, disjoint open sets G and H in P ∋: D ⊆ G and E ⊆ H. Seeing that G and H are open in P, Q ∩ G and Q ∩ H are open in Q. Also we have D ⊆ G and E ⊆ H implies Q ∩ D ⊆ Q ∩ G, Q ∩ E ⊆ Q ∩ H. So D ⊆ Q ∩ G and E ⊆ Q ∩ H and (Q ∩ G) ∩ (Q ∩ H) = Q ∩ (G∩ H) = ϕ. Hence Q is ws-normal.

**Theorem 8.4.6:** If P is normal and Tws–space, then P is ws-normal.

**Proof:** Take up P be a normal space. Assume D and E be a pair of disjoint ws-closed sets in

P. Seeing that Tws–space, D and E are closed sets in P. Also P normal, ∃ a pair

of disjoint open sets G and H in P ∋: D ⊆ G and E ⊆ H. Hence P is ws-normal.

**Theorem 8.4.7:** Each ws-normal space is w-normal.

**Proof:** Take up P be a ws-normal space. Assume D and E be a pair of disjoint w-closed sets in P. D and E are ws-closed sets in P. seeing that P is ws-normal, ∃ a pair of disjoint open sets G and H in P ∋: D ⊆ G and E ⊆ H. Hence P is w-normal.

**Theorem 8.4.8:** Map h: P → Q is bijective, open, ws-irresolute from a ws-normal space P onto Q then is ws-normal.

**Proof:** Take up D and E be disjoint ws-closed sets in Q. Thus h –1(D) and h –1(E) are disjoint ws-closed sets in P as h is ws-irresolute. Seeing that P is ws-normal, disjoint open sets G and H in P ∋: h –1(D) ⊆ G and h –1(E) ⊆ H. As h is bijective and open, h(G) and h(H) are disjoint open sets in Q ∋: D ⊆ h(G) and E ⊆ h(H). Hence Q is ws-normal.

**Theorem 8.4.9**: The following statements for a space P are one and the same.:

1. P is ws-normal.
2. ws-closed set D and each ws-open set K ∋: D ⊆ M, ∃an open set N ∋: D ⊆ L ⊆ cl(L ) ⊆ M.
3. For any disjoint ws-closed sets D, E, ∃an open set L ∋: D ⊆ L and cl(L ) ∩ E = Φ.
4. For each pair D, E of disjoint ws-closed sets there exist open sets M and L ∋:

D ⊆ M, E ⊆ L and cl (M) ∩ cl (L) = Φ.

**Proof:**

(1) ⟹ (2): Take D be a ws-closed set and M be a ws-open set ∋: D ⊆ M.

Then D and P –M are disjoint ws-closed sets in P. Seeing that P is ws-normal, ∃a pair of disjoint open sets L and W in P ∋: D ⊆ L and P –M ⊆ W.

Now P –W ⊆ P – (P –M), so P –W ⊆ M also L ∩W = Φ implies L ⊆ P –W, so cl (L) ⊆ cl(P –W) which implies cl(L ) ⊆ P – W. Henceforth cl (L) ⊆ P –W ⊆ M. So cl (L) ⊆ M. Hence D ⊆ L ⊆ cl (L) ⊆ M.

(2) ⟹ (3): Let D and E be a pair of disjoint ws-closed sets in P.

Now D∩E = Φ, so D⊆ P –E, where D is ws-closed and P – E is ws-open. Then by (ii) ∃an open set L ∋: D ⊆ L ⊆ cl (L) ⊆ P – E. Now cl (L) ⊆ P –E implies cl (L) ∩ E =Φ. Thus D ⊆ L and cl (L) ∩ E = Φ

(3) ⟹ (4): Let D and E be a pair of disjoint ws-closed sets in P. Then from (iii) ∃an open set M ∋: D ⊆ M and cl (M) ∩E = Φ. Seeing that cl (L) is closed, so ws-closed set. Therefore cl (L) and E are disjoint ws-closed sets in P. By hypothesis, ∃an open set L, ∋: E ⊆ L and cl (M) ∩ cl (L) = Φ.

(4)⟹ (1): Let D and E be a pair of disjoint ws-closed sets in P. Then from (iv) an open sets M and L in P ∋: D ⊆ M, E ⊆ L and cl (M) ∩cl (L) = Φ. So D ⊆ E ⊆ L and M ∩ L = Φ. Hence P ws-normal.

**Theorem 8.4.10**: Let P be a topological space. Then the following are one and the same.:

1. P is normal.
2. For any disjoint closed sets D and E, disjoint ws-open sets M and L ∋: D ⊆ M, E ⊆ L.
3. For any closed set D and any open set L ∋: D ⊆ L , ∃an ws-open set M of P ∋: D ⊆ M ⊆ α-cl(M) ⊆ L.

**Proof:**

(1) ⟹ (2): Suppose P is normal. Seeing that each open set is ws-open [ ], (ii) follows.

(2) ⟹ (3): Suppose (ii) holds. Let D be a closed set and L be an open set containing

D. Then D and P –L are disjoint closed sets. By (ii), disjoint ws-open sets

M and W ∋: D⊆ M and P–L ⊆ W, seeing that P –L is closed, so ws-closed. From Theorem 2.3.14 [2], we have P –L ⊆ αint (W) and M ∩ αint (W) = Φ and so we have

Cl (M) ∩ αint (W) = Φ. Hence D ⊆ M ⊆ α-cl(M) ⊆ P –αint(W) ⊆ L. Thus D⊆M ⊆ α-cl (M) ⊆ L.

(3) ⟹ (1): Let D and E be a pair of disjoint closed sets of P. Then A ⊆ P–E and P–E is open. ∃a ws-open set G of P ∋: D ⊆ G ⊆ α-cl (G) ⊆ P–E. Seeing that D is closed, it is ws- closed, we have D ⊆ int (G). Take M = int (cl (int (αint (G)))) and

L = int (cl (int(P – αcl (G)))).Then M and L are disjoint open sets of P ∋: A ⊆ M and

E ⊆ L. Hence P is normal.